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BISPECTRA OF INTERNAL WAVES

by

C. H. McComas III

March 1978

TECHNICAL REPORT

Prepared for the National Science Foundation under Grant OCE76-23532 and in part by Grants OCE77-25803 and OCE76-14739.

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C. H. McComas III†

†INSTITUTE OF GEOPHYSICS AND PLANETARY PHYSICS
UNIVERSITY OF CALIFORNIA AT SAN DIEGO
LA JOLLA, CALIFORNIA 92093

Presently at

WOODS HOLE OCEANOGRAPHIC INSTITUTION
Woods Hole, Massachusetts 02543

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Abstract

Derived from the equations of motion, the bispectrum of power indicates the rate of energy transfer among components of the internal wave field. This, or any other bispectrum, can be evaluated from weak resonant interaction theory given the wave spectrum. Using the Garrett and Munk model of the deep open ocean internal wave spectrum, the bispectrum of power and the closely related auto-bispectrum of vertical displacements are evaluated numerically with the intention of providing an observational test of the weak interaction theory and its predictions. The resulting levels of the bispectra for typical deep ocean internal waves are generally too low to be observed with any statistical confidence in an experiment of reasonable length and cost.

I. Introduction

As a first approximation, the oceanic internal wave field may be regarded as a superposition of independently propagating, dispersive, linear free waves. The frequency is related to the wave number through a dispersion relation. The statistics of such a field are Gaussian; all odd moments are zero (zero mean assumed) and all even moments may be expressed as products of second moments. Hence the wave field may be completely described by its three dimensional energy spectrum.

Under some further simplifying assumptions, Garrett and Munk (1972, 1975), henceforth GM, sought to construct a model spectrum consistent with the available observational evidence. Wunsch (1976) has found that this model describes the spectral shape and general energy level in much of the ocean; even in regions not far removed from various (suspected) sources of internal wave energy input. This first approximation has proved to be both a useful and reliable description of the space and time scales of the internal wave field.

However, the linear model cannot explain the apparently universal shape and level of the spectrum over a wide variety of input mechanisms. Nor can a linear field transfer energy from the (probably) large input scales (c.f. Müller 1977, Watson, West, and Cohen 1977, Bell 1977) to the dissipation occurring at small scales. Although the kinematics of the wave field are well described by the linear approximation, the dynamics are contained in the inherent nonlinearity of the motion.

If the wave amplitude is sufficiently small, one may include the nonlinear dynamics, yet retain the kinematics of the linear approximation. The small wave amplitude is expanded in a perturbation series: the first order corresponds to the linear approximation. At the next order, a linear equation for the second order motion is forced by products of two first order waves. If the amplitude is small enough so that the energy transfer times are long compared to the period, only resonant waves interact significantly. Because resonance requires the forced wave to obey the dispersion relation, the second order waves have precisely the same form as the first order waves. The interaction may then be regarded as a slow transfer of energy among the first order "linear" wave components.

McComas and Bretherton (1977), henceforth MB, used weak resonant interaction theory to investigate the transfer of energy with the GM model of the internal wave spectrum. They showed that three classes of interacting triads dominated the spectral energy transfers and appeared to explain several of the observed features of the universal spectral shape. McComas (1977) suggested that these particular mechanisms maintained that shape while transferring energy to small scale dissipation. Although these results are appealing and the predicted features are observed, there has been no direct test of this theory. Furthermore, at small vertical scales ($\sim$10 m), the predicted energy transfers occur in the order of one period - violating the underlying assumption of the resonant approach. Such waves seem to interact so strongly, that

frequencies other than the resonant frequencies must be forced, and the wave field is more like turbulence than weakly interacting, nearly-linear waves. As the accuracy of the resonant interaction approach for such strong interactions is unknown, it would be desirable to have direct observations of nonlinear interactions and spectral energy transfer to test the limits of the weak interaction concept and its predictions.

As mentioned, a linear dispersive field had Gaussian statistics. In the presence of nonlinearities however, the waves are no longer independent and the statistics will not be Gaussian. In particular, third order moments and their double Fourier transform, bispectra, will no longer be zero. Hence these quantities serve as measures of the nonlinearity of the wave field.

In contrast to the routine use of second order spectra, application of third (and higher) order spectra has been sporadic, confused, and difficult to interpret, even though the basic mathematical and computational foundation is replete (c.f. Hasselmann *et al.* 1963, Haubrich 1965, Brilliger and Rosenblatt 1967, Hinich and Clay 1968). Most previous work (Huber *et al.* 1971 on EEG's, Roden and Bendinger 1973 of various oceanic variables, Neshyba and Sobey 1975 on ocean internal waves, and many others) has been confined to producing frequency or one dimensional wave number bispectra of the record at hand. While this can indicate which frequencies or wave numbers are nonlinearly coupled, it is an overly restricted approach. The full potential of bispectra, when combined with the equations of motion, is to provide a test of nonlinear dynamics and nonlinear interaction theories.

This paper concerns the use of bispectra to investigate nonlinear internal wave dynamics in the ocean. It has two main objectives: 1) to define and evaluate bispectra which can be used to directly observe the nonlinear coupling causing energy transfer within the internal wave spectrum, and 2) to use bispectra to test the adequacy of weak interaction theory as a description of that nonlinear coupling.

Toward the first objective, we show that the time rate of change of the internal wave energy spectrum at a given wave number and/or frequency is given by an integral over a certain bispectrum. This "bispectrum of power" gives the rate of energy input to a given wave component by nonlinear coupling of two other wave components. If the required observational records can be obtained, the direct determination of the waves involved in spectral redistribution and an estimate of the energy transfer rate is possible. For comparison with the observed bispectrum that might arise from such an observational program, we produce the bispectrum of power evaluated from weak resonant interaction theory. This bispectrum shows strong interactions at the wave numbers and frequencies corresponding to the interaction mechanisms noted by MB. If an observed power bispectrum shows similar peaks, it would provide solid confirmation of those theoretical mechanisms (objective 2).

Any bispectrum provides some measure of nonlinear coupling. Although the bispectrum of power is the best measure of nonlinear energy transfer, the required observational records will be difficult to obtain.

Some more readily observed Lagrangian bispectra can be evaluated from weak interaction theory. It results that the auto-bispectrum of vertical displacements of a fluid particle is a good test of the elastic scattering interaction mechanism (MB, McComas 1977) responsible for the equal intensities of up- and down-going waves (vertically symmetric spectra) often observed in the deep ocean. This bispectrum is also evaluated numerically.

Unfortunately, the theoretical results indicate that bispectra are rather insensitive observational tests of the nonlinear interactions in the internal wave field in that their low levels require a large number of observations to attain statistical confidence. Some speculation on why this is so is given. The extremely low level of the frequency auto-bispectrum of vertical displacements is particularly surprising since Neshyba and Sobey (1975) have reported highly significant levels. It is suggested that their positive result is due to the method of sampling, not to strong interactions of the waves themselves.

In the next section, some background on the definition, interpretation, interrelations, and display of various bispectra is given. In section 3 the important bispectrum of power is developed directly from the energy equation for internal waves. Section 4 shows that this bispectrum or any other Lagrangian bispectrum can be evaluated using weak resonant interaction theory. In section 5, the bispectrum of

power and the auto-bispectrum of vertical displacement are evaluated using the result of section 4. Some suggestions as to an observational program and a summary are contained in the final section.

2. Higher Order Spectra: Backgound

Higher order spectra, like their second order counterparts, are useful analytical tools. It is the equations of motion and our knowledge of the phenomenon and its dynamics which leads to the proper use and interpretation of any spectral analysis. This section is a brief background on the definition and interpretation of third order spectra.

2.1 Definitions

As third order spectra are in many ways similar to the familiar second order spectra, it is helpful to approach the higher order spectra by analogy. The cross-spectrum of two stationary homogeneous random functions $u(\underline{x},t)$, $v(\underline{x},t)$ of position and time is defined as

$$G_{uv}(\underline{k},\omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} R_{uv}(\underline{r},\tau)\, e^{-i\underline{k}\cdot\underline{r}-i\omega\tau} d\underline{r}d\tau \quad , \tag{2.1}$$

where

$$R_{uv}(\underline{r},\tau) = \langle u(\underline{x},t)v(\underline{x}+\underline{r},t+\tau)\rangle \quad . \tag{2.2}$$

The bracket $\langle\ \rangle$ denotes ensemble means. The cross-bispectrum of three stationary homogeneous random functions $u(\underline{x},t)$, $v(\underline{x},t)$, $w(\underline{x},t)$ is

$$B_{uvw}(\underline{k}',\omega',\underline{k}'',\omega'') = \frac{1}{(2\pi)^8} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \tag{2.3}$$

$$T_{uvw}(\underline{r}',\tau',\underline{r}'',\tau'')e^{-i\underline{k}'\cdot\underline{r}'-i\omega'\tau'-i\underline{k}''\cdot\underline{r}''-i\omega''\tau''} d\underline{r}'d\underline{r}''d\tau'd\tau''$$

where

$$T_{uvw}(\underline{r}',\tau',\underline{r}'',\tau'') = \langle u(\underline{x},t)v(\underline{x}+\underline{r}',t+\tau')w(\underline{x}+\underline{r}'',t+\tau'')\rangle \quad . \tag{2.4}$$

The inverse relations to (2.1) and (2.3) are

$$R_{uv}(\underline{r},\tau) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} G_{uv}(\underline{k},\omega)e^{i\underline{k}\cdot\underline{r}+i\omega\tau}d\underline{k}d\omega \tag{2.5}$$

and

$$T_{uvw}(\underline{r}',\tau',\underline{r}'',\tau'') = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} B_{uvw}(\underline{k}',\omega',\underline{k}'',\omega'')$$
$$e^{i\underline{k}'\cdot\underline{r}'+i\omega'\tau'+i\underline{k}''\cdot\underline{r}''+i\omega''\tau''}d\underline{k}'d\underline{k}''d\omega'd\omega'' \quad . \tag{2.6}$$

As noted by Hasselmann *et al.* (1963) the interpretation of these cross-spectra is more readily apparent in terms of the components $dU(\underline{k},\omega)$ of a Fourier-Stieltjes representation:

$$u(\underline{x},t) = \int_{-\infty}^{\infty} dU(\underline{k},\omega)e^{i\underline{k}\cdot\underline{x}+i\omega t} \tag{2.7}$$

so that

$$\langle dU(\underline{k}',\omega')dV(\underline{k}'',\omega'')\rangle =$$

$$\begin{cases} G_{uv}(\underline{k}',\omega')d\underline{k}'d\omega' & \text{if} \quad \omega' = \omega'' \quad \text{and} \quad \underline{k}' = \underline{k}'' \\ 0 & \text{if not} \end{cases} \tag{2.8}$$

and

$$\langle dU(\underline{k},\omega)\,dV(\underline{k}',\omega')\,dW(\underline{k}'',\omega'')\rangle \;=$$

$$\begin{cases} B_{uvw}(\underline{k}',\omega',\underline{k}'',\omega'')\,d\underline{k}'\,d\omega'\,d\underline{k}''\,d\omega'' & \text{if} \quad \omega'+\omega''+\omega = 0 \\ & \text{and} \quad \underline{k}'+\underline{k}''+\underline{k} = 0 \\ 0 & \text{if not} \end{cases} \tag{2.9}$$

The cross-spectrum G_{uv} represents the contribution to $\langle uv \rangle$ from the product of two Fourier components whose frequencies and wave numbers add to zero, whereas B_{uvw} represents the contribution to $\langle uvw \rangle$ from the product of three Fourier components whose frequencies and wave numbers add to zero. (The latter condition is <u>not</u> the same as resonance, because the frequencies are not restricted to satisfy the dispersion relation. Nonlinearities may force waves at frequencies other than the natural frequency for the resultant wave number, and (2.9) includes these forced, non-resonant waves.) If the statistics are not homogeneous and/or stationary, this interpretation for the bispectrum is not valid. Those processes which are not homogeneous should use frequency decompositions at a point, and non-stationary processes should use wave number decompositions at an instant of time.

2.2 Symmetries:

Under certain conditions third order correlations and their associated bispectra show certain symmetries similar to those of second order spectra. If $u(\underline{x},t)$, $v(\underline{x},t)$, $w(\underline{x},t)$ are real,

$$G_{uv}(\underline{k},\omega) = G^{*}_{uv}(-\underline{k},-\omega) \tag{2.10}$$

$$B_{uvw}(\underline{k}',\omega',\underline{k}'',\omega'') = B^{*}_{uvw}(-\underline{k}',-\omega',-\underline{k}'',-\omega'') \quad . \tag{2.11}$$

For stationary and homogeneous processes

$$R_{uv}(\underline{r},\tau) = R_{vu}(-\underline{r},-\tau) \tag{2.12}$$

and

$$\begin{aligned} T_{uvw}(\underline{r}',\tau',\underline{r}'',\tau'') &= T_{uwv}(\underline{r}'',\tau'',\underline{r}',\tau') \\ &= T_{vuw}(-\underline{r}',-\tau',\underline{r}''-\underline{r}',\tau''-\tau') \\ &= T_{vwu}(\underline{r}''-\underline{r}',\tau''-\tau',-\underline{r}',-\tau') \\ &= T_{wvu}(\underline{r}'-\underline{r}'',\tau'-\tau'',-\underline{r}'',-\tau'') \\ &= T_{wuv}(-\underline{r}'',-\tau'',\underline{r}'-\underline{r}'',\tau'-\tau'') \quad . \end{aligned} \tag{2.13}$$

(The first relation in (2.13) is true without stationarity or homogeneity.)

From these relations

$$G_{uv}(\underline{k},\omega) = G_{vu}(-\underline{k},-\omega) \tag{2.14}$$

and

$$\begin{aligned} B_{uvw}(\underline{k}',\omega',\underline{k}'',\omega'') &= B_{uwv}(\underline{k}'',\omega'',\underline{k}',\omega') \\ &= B_{vuw}(\underline{k},\omega,\underline{k}',\omega') \\ &= B_{vwu}(\underline{k}',\omega',\underline{k},\omega) \\ &= B_{wvu}(\underline{k},\omega,\underline{k}'',\omega'') \\ &= B_{wuv}(\underline{k}'',\omega''\underline{k},\omega) \end{aligned} \tag{2.15}$$

where $\underline{k} = -\underline{k}'-\underline{k}''$ and $\omega = -\omega'-\omega''$

2.3 Display

Because there are a large number of possible symmetries, a standard format for display, at least for third order spectra of one space dimension or time, will prove useful. The following is not unique, but is relatively simple.

The first task is to choose an order for the placement of the component records. If all three records u,v,w are different this is arbitrary. Having chosen say uvw the frequency cross-bispectrum $B_{uvw}(\omega',\omega'')$ can be displayed in the 2-space (ω',ω''). See Figure 1. This bispectrum represents the product of Fourier components ω' from record v, ω'' from w, and $\omega = -\omega' - \omega''$ from u.

Because of the reality condition (2.11) there is a symmetry about the origin such that $B_{uvw}(\omega',\omega'')$ is unique in any half plane of (ω',ω''). The half plane $\omega' > 0$, octants 1, 2, 7, and 8 of Fig. 1 is a convenient choice. The entire plane may be reconstructed by reflection through the origin.

If any two records are the same, i.e. u, v, u, placing the odd record first, so that the display frequencies ω', ω'' come from the same record, introduces the simple symmetry

$$B_{vuu}(\omega',\omega'') = B_{vuu}(\omega'',\omega') \quad . \tag{2.16}$$

Thus, this spectrum is unique in the quarter plane

$$\omega' \geq 0 \quad , \quad \omega' \geq |\omega''| \tag{2.17}$$

octants 1 and 8. With this additional symmetry, the entire plane is obtained by reflecting the unique quarter plane about the folding axes of Fig. 1, as one would unfold a quartered piece of paper.

If all three records are the same, i.e.. u, u, u the symmetry

$$B_{uuu}(\omega',\omega'') = B_{uuu}(\omega,\omega'') \tag{2.18}$$

implies uniqueness in octant 1 where $\omega' \geq \omega'' \geq 0$. Unfortunately no simple folding symmetry is apparent and (2.18) must be used explicitly to obtain the quarter plane, from which the entire plane can be gained as before.

2.4 Eulerian versus Lagrangian Bispectra:

Any flow field may be regarded as either a function of time and fixed position in space or as a function of time and the position of a particle. The Eulerian and Lagrangian representations are completely equivalent descriptions of the field, and the choice of either system is a matter of convenience.

Although at any one point at any given time, a Lagrangian or Eulerian variable must have precisely the same value, a time history of a Lagrangian variable is different from the time history of an Eulerian variable, as they are indeed records of quite different quantities. This raises the question of what relationship, if any, Eulerian frequency spectra have to Lagrangian frequency spectra.

An Eulerian variable, $Q_E(\underline{x},t)$, forever at position $\underline{x}$ can be related to the Lagrangian variable, $Q_L(\underline{x},t)$, whose initial position was $\underline{x}$ through the Lagrangian displacement $\underline{\xi}_L(\underline{x},t)$ by

$$Q_L(\underline{x},t) = Q_E(\underline{x},t) - \left[\xi_L(\underline{x},t)\right]_i \frac{\partial Q_L(\underline{x},t)}{\partial x_i} + \cdots \qquad (2.19)$$

correct to second order (Phillips, 1966). Clearly, second order time correlations of Lagrangian and Eulerian variables are equivalent to second order in the wave amplitude, i.e.

$$\langle Q_E Q_E \rangle = \langle Q_L Q_L \rangle - \langle \xi_{L_i} \frac{\partial Q_L}{\partial x_i} Q_L \rangle - \langle Q_L \xi_{L_i} \frac{\partial Q_L}{\partial x_i} \rangle + \cdots \tag{2.20}$$

and if the wave motion is of small amplitude, or the statistics nearly Gaussian, the Eulerian and Lagrangian second order products and their related spectra are essentially interchangeable.

Unfortunately, the same is not always true for third order time correlations and frequency bispectra, as

$$\langle Q_E Q_E Q_E \rangle = \langle Q_L Q_L Q_L \rangle - \langle \xi_{L_i} \frac{\partial Q_L}{\partial x_i} Q_L Q_L \rangle - \langle Q_L \xi_{L_i} \frac{\partial Q_L}{\partial x_i} Q_L \rangle - \langle Q_L Q_L \xi_{L_i} \frac{\partial Q_L}{\partial x_i} \rangle + \cdots \tag{2.21}$$

Again, if the wave motion is of small amplitude, the two third order products are nearly equal - unless the statistics are Gaussian to first order. In that case, the first order of the triple product disappears, making all the terms on the right hand side of (2.21) comparable in magnitude. Hence, Lagrangian and Eulerian triple products and their bispectra can be quite different.

2.5 Interrelationships:

For a homogeneous and stationary field,

$$B_{uvw}(\omega',\omega'') = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} B_{uvw}(\underline{k}',\underline{k}'',\omega',\omega'')\ d\underline{k}'d\underline{k}'' \tag{2.22}$$

$$B_{uvw}(\underline{k}',\underline{k}'') = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} B_{uvw}(\underline{k}',\underline{k}'',\omega',\omega'')\ d\omega'd\omega'' \tag{2.23}$$

$$B_{uvw}(k_1',k_2'') = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} B_{uvw}(\underline{k}',\underline{k}'',\omega',\omega'')\ dk_2'dk_3''\ dk_1''dk_3''d\omega_1 d\omega_2 \tag{2.24}$$

etcetera where $\underline{k} = (k_1,k_2,k_3)$. Any bispectral density of some subset of arguments is an integral of the full bispectral density $B_{uvw}(\underline{k}',\underline{k}'', \omega',\omega'')$ over the remaining arguments.

Because Lagrangian velocities are related to the associated Lagrangian displacements through the frequency, any Lagrangian frequency bispectrum involving velocities can be obtained from the displacement bispectrum and vice versa, for example

$$B_{u_3u_3u_3}(\omega',\omega'') = i\omega'(i\omega'')(i\omega)\ B_{\xi_3\xi_3\xi_3}(\omega',\omega'') \quad . \tag{2.25}$$

This results because the vertical velocity, u_3, is linearly related to the vertical displacement ξ_3, so that their Fourier transforms are simply related by the frequency, i.e.,

$$dU_3(\omega) = i\omega \, d\xi_3(\omega) \quad . \tag{2.26}$$

The same degeneracy is not true of Eulerian bispectra because an Eulerian velocity is nonlinearly related to the displacements.

An Eulerian or Lagrangian wave number bispectrum containing gradients can be related to the corresponding wave number bispectrum without gradients, e.g.,

$$B_{uv\frac{dw}{dx_j}}(\underline{k}',\underline{k}'') = ik_j'' \, B_{uvw}(\underline{k}',\underline{k}'') \tag{2.27}$$

The only restriction is that the gradient must have the corresponding wave number as an argument of the bispectral density. Clearly, frequency bispectra of gradients and wave number bispectra of velocities cannot be simply related to their respective displacement spectra.

2.6 Interpretation of Bispectra

As second order spectra are harmonic analyses of second moments, third order spectra are harmonic analyses of third moments. For stationary and homogeneous fields they each have the simple interpretations in (2.8) and (2.9). However, bispectra, unlike second order spectra, are exactly zero at all wave numbers and/or frequencies if the statistics are Gaussian (Haubrich 1965). When three wave numbers and/or frequencies are nonlinearly coupled, the bispectrum has non-zero values at those wave number and/or frequency points.

What does a particular bispectral value mean? Formally, it is the contribution from the region $\Delta\underline{k}'\Delta\underline{k}''$ and/or $\Delta\omega'\Delta\omega''$ to that particular triple moment. It identifies those waves which contribute most to that moment. The significance of that moment must be obtained from the equations of motion. Waves producing a large bispectral value for one triple moment may or may not have large bispectral values for another moment.

In general, a given bispectral value may arise in two ways: either by weak coupling of highly "energetic" waves or by strong coupling of low intensity waves. Therefore the bispectrum itself is not a good indicator of the degree of coupling.

2.7 Bicoherence

It has been suggested that the "bicoherence" defined by

$$C_{uvw}\ (\underline{k}',\omega',\underline{k}'',\omega'';\Delta\underline{k}',\Delta\omega',\Delta\underline{k}'',\Delta\omega'') \equiv$$

$$\frac{|B_{uvw}(\underline{k}',\underline{k}'',\omega',\omega'')\Delta\underline{k}'\Delta\underline{k}''\Delta\omega'\Delta\omega''|}{\left[P_u(\underline{k},\omega)\Delta\underline{k}\Delta\omega\ P_v(\underline{k}',\omega')\ \Delta\underline{k}'\Delta\omega'\ P_w(\underline{k}'',\omega'')\Delta\underline{k}''\Delta\omega''\right]^{\frac{1}{2}}} \qquad (2.28)$$

where $\langle uu\rangle = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} P_u(\underline{k},\omega)\,d\underline{k}d\omega$ (2.29)

is an appropriate measure of coupling strength (Hinich and Clay, 1968). It is zero when there is no coupling, but unfortunately it is not independent of bandwidth (c.f. Hasselmann *et al.* 1963). However, for a fixed bandwidth in a given bispectrum of a single process, it should be a measure of the relative strength of coupling, e.g., waves 1, 2, and 3 are more strongly coupled than 4, 5 and 6 because 1, 2, and 3 has a larger bispectral value for the amount of energy contained in those three waves. Computations from atmospheric and wind-tunnel data, although yielding greatly different bispectral levels, indicate a rather constant coupling coefficient (Helland *et al* 1977A).

The bicoherence C does not integrate like the bispectrum does, i.e.

$$C_{uvw}(\omega',\omega'';\Delta\omega',\Delta\omega'') \equiv \frac{|B_{uvw}(\omega',\omega'')\Delta\omega'\Delta\omega''|}{\left[P_u(\omega')\Delta\omega'\ P_v(\omega')\Delta\omega'\ P_w(\omega'')\Delta\omega''\right]^{\frac{1}{2}}} \tag{2.30}$$

$$\neq \sum_{\underline{k}'} \sum_{\underline{k}''} C_{uvw}(\underline{k}',\omega',\underline{k}'',\omega'';\Delta\underline{k}',\Delta\omega',\Delta\underline{k}'',\Delta\omega'')$$

Thus for internal gravity waves, whose frequency is independent of scale, strongly coupled small scale waves might not be detected by the bicoherence defined in (2.30), because the bispectral level of the small scale interaction would be normalized by the total energy in the frequency band, not just the much smaller energy of only the small scale waves. The same type of masking could occur for wave number bicoherences $C(\underline{k}',\underline{k}'';\Delta k',\Delta k'')$, which lump the less energetic high frequencies in with the low frequencies.

In general, the interpretation of bicoherences of the form (2.30) must be made with care. A low value indicates that the energetic scales are not coupled strongly, but says nothing about the coupling of the low energy scales. A high value indicates strong coupling within the band, but does not indicate which scales within the band produce that coupling.

Because the variance of an observed bispectrum is proportional to the product of the power in each wave band (Rosenblatt and Van Ness 1965), these bicoherences are useful in designing experiments. From the theoretically determined bispectral level, the length of record required to reliably observe that level can be estimated from the bicoherence. Because of possible masking difficulties, this is the only manner in which bicoherence will be used in this paper.

3. Bispectra and Internal Wave Energy Dynamics

We have seen that spectra show correlations between two frequencies and/or wave numbers, while bispectra show correlations between three frequencies and/or wave numbers. Thus, auto-spectra of a stationary and homogeneous process would indicate that a frequency or wave number is correlated only with itself. At this level, second order spectra would seem rather uninformative. But, second order spectra also indicate the total variance within each band, and we often find certain combinations of spectra, such as energy, are dynamically revealing. Similarly, third order spectra, while automatically yielding interesting information on the nature and strength of any nonlinear coupling, are also important in nonlinear dynamics and the redistribution (and dissipation) of energy. In this section, we will show that for stationary and homogeneous processes, integrals of particular bispectra give the time rate of change of the energy spectrum. This interpretation is obtained directly from the energy equation. The approach follows previous treatments of turbulence spectra (c.f. in Phillips 1966) and bispectral analysis (Lii *et al.*, 1976, Hellund *et al.*, 1977B).

The equations of motion for an adiabatic, inviscid, incompressible, uniformly rotating, stratified, Boussinesq fluid are

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \varepsilon_{ijk} f_j u_k + \frac{\partial p}{\partial x_i} - b\delta_{i3} = 0 \qquad (3.1)$$

$$\frac{\partial b}{\partial t} + u_j \frac{\partial b}{\partial x_j} = -N^2 u_j \delta_{j3} \qquad (3.2)$$

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{3.3}$$

$$\rho(\underline{x},t) = \rho_o + \bar{\rho}(x_3) + \rho'(\underline{x},t) \tag{3.4}$$

$$\frac{\partial \bar{p}(x_3)}{\partial x_3} = -\left(1 + \frac{\bar{\rho}(x_3)}{\rho_o}\right) g \tag{3.5}$$

where $\underline{u}(\underline{x},t)$ is the velocity perturbation, $\underline{f}$ is the rotation vector, $p(\underline{x},t)$ is the reduced kinematic pressure, $b(\underline{x},t)$ is the buoyancy perturbation, and $N^2(x_3)$, the Brunt-Väisälä frequency, is a function of vertical position only. The basic state is at rest with the mean pressure, $\bar{p}(x_3)$, in hydrostatic balance.

The energy equation for this motion

$$\frac{\partial}{\partial t} E(\underline{x},t) + \frac{\partial}{\partial x_j} F_j(\underline{x},t) = S(\underline{x},t) \tag{3.6}$$

relates the time rate of change of total energy

$$E(\underline{x},t) = \tfrac{1}{2} \langle u_i u_i + \frac{1}{N^2} bb \rangle \tag{3.7}$$

a divergence of a flux

$$F_j(\underline{x},t) = \langle p u_j \rangle \tag{3.8}$$

and an energy source term

$$S(\underline{x},t) = \left\langle u_i u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{N^2} b u_j \frac{\partial b}{\partial x_j} \right\rangle \tag{3.9}$$

$$= \left\langle u_i u_j e_{ij} + \frac{1}{N^2} b u_j \frac{\partial b}{\partial x_j} \right\rangle$$

where $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the rate of strain tensor. For imcompressible flows, this energy source term is equivalent to a divergence of an energy flux,

$$\mathcal{F}_j(\underline{x},t) = \left\langle \tfrac{1}{2} u_j \left(u_i u_i + \frac{1}{N^2} bb\right) \right\rangle \tag{3.10}$$

due to self advection.

If the statistics are homogeneous, the time rate of change of energy at a point is exactly zero, as all fluxes are required to be independent of position. No energy builds up at a point. There may still be a linear or nonlinear flux in physical space, but that flux is non-divergent. Similarly, the nonlinear source term may cause transfer between wave numbers, but when summed over all wave numbers, this transfer must be zero. For oceanic internal waves, the assumption of homogeneity is valid only in the WKBJ limit of scales of motion small compared to the scale of background inhomogeneity, the vertical variation of the Brunt-Väisälä frequency.

The question of stationarity is more difficult. For linear processes, an initially non-Gaussian field tends rather rapidly to a Gaussian state (Hasselmann, 1967), in which all third order moments,

correlations, and spectra are exactly zero. With nonlinearities, third order moments, whose rate of growth depend on third and fourth order moments, arise - even from an initially Gaussian distribution. For infinitely weak nonlinear interaction, where the only contributions come from resonant interactions, third order moments arising from an initially Gaussian distribution tend to a stationary value (section 4). However, stationarity is not possible when the nonlinear interactions are strong and the frequency interactions are promiscuous. An assumption of a stationary bispectrum is equivalent to an assumption of weak nonlinearity.

Bispectra of weakly nonlinear fields are more sensitive to intermittency (and possible stationarity) than are second order spectra. Bursts of nonlinear activity are just small perturbations to the otherwise constant second order spectra which are largely determined by the Gaussian linear field. But, because the bispectrum reflects only the nonlinear activity - the Gaussian linear field not contributing to the bispectrum - fluctuations in nonlinear activity may cause major changes in the bispectral level. The presence of the linear field also complicates the determination of the bispectrum, as the "signal" (the nonlinear coupling) must be extracted from the large "noise" of the linear field.

when using third order spectra, as in using second order spectra, the underlying assumptions of stationarity or homogeneity should be carefully examined.

Granted stationarity on time scales the length of the record and homogeneity on space scales assumed small compared to the scale of variation of the Brunt-Väisälä frequency, then if

$$u_i(\underline{x},t) = \int e^{i\underline{k}\cdot\underline{x}+i\omega t}\, dU_j(\underline{k},\omega)$$

$$b(\underline{x},t) = \int e^{i\underline{k}\cdot\underline{x}+i\omega t}\, dB(\underline{k},\omega) \tag{3.11}$$

$$p(\underline{x},t) = \int e^{i\underline{k}\cdot\underline{x}+i\omega t}\, dP(\underline{k},\omega) \quad ,$$

we obtain from (3.1)

$$\frac{\partial}{\partial t}\, dU_j(-\underline{k},-\omega) + \mathcal{E}_{j\ell n}\, f_\ell\, dU_n(-\underline{k},-\omega) - ik_j dP(-\underline{k},-\omega) - \delta_{j3} dB(-\underline{k},-\omega)$$

$$= -\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[ik''_\ell dU_\ell(\underline{k}',\omega')\, dU_j(\underline{k}'',\omega'') + ik'_\ell dU_\ell(\underline{k}'',\omega'')\, dU_j(\underline{k}',\omega')\right]$$

$$e^{i(\underline{k}'+\underline{k}''+\underline{k})\cdot\underline{x} + i(\omega'+\omega''+\omega)t}\, d\underline{k}'\, d\underline{k}''\, d\omega'\, d\omega'' \quad . \tag{3.12}$$

Using a similar equation for buoyancy obtained from (3.2), multiplying each equation appropriately, and combining to obtain the time rate of change of the energy spectrum of a homogeneous and stationary process yields

$$\frac{\partial}{\partial t}\hat{E}(\underline{k},\omega) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{Re}\left[\hat{S}(\underline{k}',\omega',\underline{k}'',\omega'')\right]\delta(\underline{k}'+\underline{k}''+\underline{k})\,\delta(\omega'+\omega''+\omega)\;d\underline{k}'d\underline{k}''d\omega'd\omega'' \tag{3.13}$$

where

$$\hat{E}(\underline{k},\omega) = \tfrac{1}{2}\left\{dU_j(\underline{k},\omega)\,dU_j(-\underline{k},-\omega) + \frac{1}{N^2}\,dB(\underline{k},\omega)\,dB(-\underline{k},\omega)\right\} \tag{3.14}$$

is the energy spectrum and

$$\hat{S}(\underline{k}',\omega',\underline{k}'',\omega'')\;d\underline{k}'d\underline{k}''d\omega'd\omega'' \equiv$$

$$\begin{cases} \tfrac{1}{2}\Big\langle -ik_j''\Big\{dU_i(\underline{k},\omega)\,dU_j(\underline{k}',\omega')\,dU_i(\underline{k}'',\omega'') \\ \qquad + \dfrac{1}{N^2}\,dB(\underline{k},\omega)\,dU_j(\underline{k}',\omega')\,dB(\underline{k}'',\omega'')\Big\} \\ -ik_j'\Big\{dU_i(\underline{k},\omega)\,dU_j(\underline{k}'',\omega'')\,dU_i(\underline{k}',\omega') \\ \qquad + \dfrac{1}{N^2}\,dB(\underline{k},\omega)\,dU_j(\underline{k}'',\omega'')\,dB(\underline{k}',\omega')\Big\}\Big\rangle & \text{if} \quad \begin{matrix}\underline{k}'+\underline{k}''+\underline{k} = 0 \\ \text{and} \\ \omega'+\omega''+\omega = 0\end{matrix} \\ 0 & \text{if not} \end{cases} \tag{3.15}$$

is the bispectrum of power. As in the energy equation (3.6), rotation has dropped out and for a homogeneous process the flux term

$$\left\langle \frac{\partial p}{\partial x_j}\,u_j \right\rangle = ik_j\;dP(\underline{k},\omega)\;dU_j(-\underline{k},\omega) \tag{3.16}$$

is also zero, as continuity requires

$$k_j \, dU_j(\underline{k},\omega) = 0 \quad . \tag{3.17}$$

(The velocity and pressure gradient of any one component are always perpendicular.) Thus, the real part of the bispectrum of power, $\hat{S}$, may be interpreted as the contribution to the rate of change of the energy spectrum at $(\underline{k},\omega)$ by the nonlinear interaction of any two waves $(\underline{k}',\omega')$ and $(\underline{k}'',\omega'')$. Under the assumptions of homogeneity and stationarity, only those pairs of waves whose wave numbers and frequencies sum to the wave number and frequency of interest, i.e. $(\underline{k},\omega)$, make any contribution. The total rate of change of the energy spectrum at a given wave number and/or frequency is the integral over all such pairs. (The power bispectrum for internal waves is the same as for turbulence (Lii *et al* 1976) except that the effects of buoyancy have been included and viscosity effects ignored.)

Using continuity it may be shown from (3.13) and (3.15) that

$$\frac{\partial}{\partial t} E \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial \hat{E}(\underline{k},\omega)}{\partial t} \, d\underline{k} d\omega \quad =$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{Re} \; S[(\underline{k}',\omega',\underline{k}'',\omega'')] \; d\underline{k} d\omega \; d\underline{k}' d\omega' \; d\underline{k}'' d\omega'' = 0 \tag{3.18}$$

and the energy at a point is conserved, as we found before.

As might have been guessed directly from (3.15), $\hat{S}$ is the bispectrum of the nonlinear source term (3.9) and may be alternately defined from (2.3) using the correlation

$$\mathcal{N}(\underline{r}',\tau',\underline{r}'',\tau'') = -\langle u_i(\underline{x},t)u_j(\underline{x}+\underline{r}',t+\tau')\frac{\partial u_i(\underline{x}+\underline{r}'',t+\tau'')}{\partial x_j} + \frac{1}{N^2}b(\underline{x},t)u_j(\underline{x}+\underline{r}',t+\tau')\frac{\partial b(\underline{x}+\underline{r}'',t+\tau'')}{\partial x_j}\rangle \quad . \tag{3.19}$$

Substitution into (3.13) results in

$$\frac{\partial E(\underline{k},\omega)}{\partial t} = \frac{1}{(2\pi)^4}\int \mathcal{N}(\underline{r},\tau,\underline{r},\tau)\cos(\underline{k}\cdot\underline{r}+\omega\tau)\, d\underline{r}d\tau \quad . \tag{3.20}$$

The time rate of change of the energy spectrum is given by the real part of the Fourier transform of the correlation $N(\underline{r},\tau,\underline{r},\tau)$. This may also be derived directly from the energy equation (3.6). Because the lags are identical, the triple correlation $\mathcal{N}$ can be considered a double correlation between the records u_i and $u_j\frac{\partial u_i}{\partial x_j}$. The time rate of change of the energy spectrum may be evaluated either by (3.13) or (3.20). The bispectrum $\hat{S}$ provides more detailed information, giving the contribution from each set of components, not just the total as in (3.20).

The bispectrum $\hat{S}$ is composed of twelve separate terms. For internal gravity waves, the scales of vertical and horizontal velocity W, U are related to vertical and horizontal space scales H, L by

$$\frac{W}{U} = O\left(\frac{H}{L}\right) \quad . \tag{3.21}$$

The bulk of the internal wave energy is at low frequencies where $H/L << 1$. Thus, the only terms that a priori might be neglected are

$$WU\,\frac{\partial W}{\partial x} = WV\,\frac{\partial W}{\partial y} = WW\,\frac{\partial W}{\partial z} = O\left(\frac{H^2}{L^2}\right)\; O\left(UU\,\frac{\partial U}{\partial x}\right)$$

leaving nine terms of equal magnitude to be evaluated. But, even these three terms will be important for frequencies near the local Brunt-Väisälä frequency, where H/L is not small.

To determine $\hat{S}$ using (3.13) requires records of velocity and buoyancy suitable for Fourier decomposition in three-dimensional wave number - even when frequency bispectra are to be computed. If gradients of velocity and buoyancy can be measured, then $\hat{S}$ may be determined from (3.19).

Because of limited measurement capabilities, particularly in the space domain, it is not likely that the full bispectral density $\hat{S}(\underline{k}',\underline{k}'',\omega',\omega'')$ can be obtained. Rather some integral of the full bispectral density, such as $\hat{S}(k_3',k_3'')$ where k_3 is a vertical wave number, or $\hat{S}(\omega',\omega'')$ might be obtained. Even these will not be easy,

and a theoretical prediction of such bispectra greatly aides the experimental design as well as subsequent interpretation. The evaluation of $\hat{S}(k_3', k_3'')$ and $\hat{S}(\omega', \omega'')$ from weak resonant interaction theory is given in the following sections.

To summarize, we have noted that the time rate of change of energy at a point is balanced by the divergence of a flux and by a nonlinear energy source term. If the process is homogeneous, the energy at a point is constant. (Viscuous and other dissipation has been ignored.) However, energy may still be transferred within the spectrum. The rate of change of the energy spectrum of a homogeneous and stationary process is the Fourier transform of a third order correlation, which is equivalent to an integral over the corresponding bispectrum. If appropriate records could be obtained, the rate of change of the energy spectrum due to nonlinear interaction with two other wave components could be directly observed through this bispectrum.

4. Bispectra and Resonant Interaction

If internal waves are indeed a weakly nonlinear process, the field may be regarded as a superposition of independently propagating random waves governed by a linear equation of motion. Frequencies are related to the wave number by a dispersion relation. This base state has Gaussian statistics and hence is fully described by its constant three-dimensional energy spectrum, $E(\underline{k})$. Any bispectrum is exactly zero.

At the next order, motion will be forced by interaction of two waves of the first order field at sum and difference wave numbers and frequencies. A resonance occurs for forced waves of the natural frequency; i.e. those frequencies satisfying the dispersion relation for the forced wave number. Asymptotically, the only significant energy transfer occurs between resonant waves (Phillips 1960).

Such nonlinear coupling should be reflected in bispectra. The objective of this section is to show that weak nonlinear interaction of an initially Gaussian, homogeneous field produces a stationary, non-zero, third order product of Fourier components from which any desired Lagrangian third order correlation or bispectrum may be produced. In the next section, the bispectrum of power and the closely related, but more readily observed, auto-bispectrum of vertical displacements are evaluated.

4.1 The Linear Field

As previously noted, the internal wave field can be considered, to a first approximation, as a superposition of independently propagating linear waves. The field may then be described by the Lagrangian displacements (Olbers 1976 and McComas 1975)

$$\xi_j(\underline{x},t) = \sum_{s=\pm 1} \sum_{\underline{k}} a^s(\underline{k})\ \hat{G}^s_{\underline{k}_j}\ e^{i(\underline{k}\cdot\underline{x}-s\Omega(\underline{k})t)} \tag{4.1}$$

with wave amplitudes $a^s(\underline{k})$ where the wave number $\underline{k} = (k_1,k_2,k_3)$ has a magnitude $|\underline{k}|$ and horizontal wave number

$$\kappa = (k_1^2 + k_2^2)^{\frac{1}{2}} \tag{4.2}$$

and

$$\Omega(\underline{k}) = \left(\frac{N^2\kappa^2 + f^2k_3^2}{|\underline{k}|^2}\right)^{\frac{1}{2}} , \tag{4.3}$$

$$\hat{G}^s_{\underline{k}} = C\begin{cases} \dfrac{k_3}{\kappa^2}\left(-k_1 - \dfrac{ifk_2}{s\omega}\right) \\ \dfrac{k_3}{\kappa^2}\left(-k_2 + \dfrac{ifk_1}{s\omega}\right) \\ 1 \end{cases} \tag{4.4}$$

are the dispersion relation and displacement vector, respectively; C is a complex constant and f is the vertical component of Earth's rotation. Reality of $\xi_j(x,t)$ implies

$$a^s(\underline{k}) = \left(a^{-s}(-\underline{k})\right)^* \tag{4.5}$$

where * denotes complex conjugate. For this Gaussian field the statistics are completely described by the power spectrum

$$E(\underset{\sim}{k}) = 2\rho_o \gamma_{\underset{\sim}{k}} \sum_{s=\pm 1} a^s(\underset{\sim}{k})\, a^{-s}(-\underset{\sim}{k}) \tag{4.6}$$

where

$$\gamma_{\underline{k}} = CC^* (N^2-f^2) \frac{\omega^2}{\omega^2-f^2} \quad . \tag{4.7}$$

So far even this simple description of the internal wave field has not been obtained, as only models of the fully three-dimensional spectrum $E(\underline{k})$ are presently available.

4.2 Nonlinear resonant interaction

To consider weak nonlinearities, the Lagrangian displacement field is given by

$$\xi_j(\underset{\sim}{x},t) = \sum_{s=\pm 1} \sum_{\underset{\sim}{k}} a^s(\underset{\sim}{k},t) \quad \hat{G}^s_{\underset{\sim}{k}\,j} \quad e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x}-s\Omega(\underset{\sim}{k})t)} \tag{4.8}$$

where the wave amplitude $a(\underset{\sim}{k},t)$ is now a function of time. Expanding $a(\underset{\sim}{k},t)$ in a perturbation series, with ε a formal parameter expressing the weakness of the field,

$$a^s(\underset{\sim}{k},t) = \varepsilon a^s_1(\underset{\sim}{k},t) + \varepsilon^2 a^s_2(\underset{\sim}{k},t) + \varepsilon^3 a^s_3(\underset{\sim}{k},t) + \ldots \tag{4.9}$$

with initial conditions

$$a^s_2(\underset{\sim}{k},0) = a^s_3(\underset{\sim}{k},0) = \ldots = 0 \quad . \tag{4.10}$$

At lowest order

$$a^s_1(\underset{\sim}{k},t) = a^s_1(\underset{\sim}{k},0) \tag{4.11}$$

a constant amplitude corresponding to the linear solution (4.1) to (4.5).

Following Hasselmann (1966), the second order amplitude is

$$a_2^s(\underline{k},t) = \sum_{s'} \sum_{s''} \sum_{\underline{k}'} \sum_{\underline{k}''} - \frac{is\Omega(\underline{k})}{4\rho_o \gamma_{\underline{k}}} D_{\underline{k}'\underline{k}''-\underline{k}}^{s's''-s} \; a_1^{s'}(\underline{k}',t) a_1^{s''}(\underline{k}'',t) \delta(\underline{k}'+\underline{k}''-\underline{k})$$

$$\left[\frac{e^{i\left(s\Omega(\underline{k})-s'\Omega(\underline{k}')-s''\Omega(\underline{k}'')\right)t} - 1}{s\Omega(\underline{k})-s'\Omega(\underline{k}')-s''\Omega(\underline{k}'')} \right] \tag{4.12}$$

where

$$D_{\underset{\sim}{k}'\underset{\sim}{k}''-\underset{\sim}{k}}^{s's''-s} = -s'k_i'k_j'\pi'\left(\hat{G}_{\underset{\sim}{k}''}^{s''}\right)_i \left(\hat{G}_{-\underset{\sim}{k}}^{-s}\right)_j - s''k_i''k_j''\pi'' \left(\hat{G}_{\underset{\sim}{k}'}^{s'}\right)_i \left(\hat{G}_{-\underset{\sim}{k}}^{-s}\right)_j$$

$$+ sk_ik_j\pi\left(\hat{G}_{\underset{\sim}{k}'}^{s'}\right)_i \left(\hat{G}_{\underset{\sim}{k}''}^{s''}\right)_j \tag{4.13}$$

and $$\pi = -iC\,(N^2-f^2)\,\frac{k_3}{|\underline{k}|^2} \tag{4.14}$$

(McComas 1975).

Consider the third order cross-correlation of any variables u, v, w for a homogeneous process

$$\langle u(\underset{\sim}{x},t)v(\underset{\sim}{x}+\underset{\sim}{r}',t+\tau')w(\underset{\sim}{x}+\underset{\sim}{r}'',t+\tau'')\rangle \;=$$

$$\sum_{\underset{\sim}{k}}\sum_{\underset{\sim}{k}'}\sum_{\underset{\sim}{k}''}\sum_{s}\sum_{s'}\sum_{s''}\hat{U}^{s}_{\underset{\sim}{k}}\hat{V}^{s'}_{\underset{\sim}{k}'}\hat{W}^{s''}_{\underset{\sim}{k}''}\langle a^{s}(\underset{\sim}{k},t)a^{s'}(\underset{\sim}{k}',t+\tau')a^{s''}(\underset{\sim}{k}'',t+\tau'')\rangle$$

$$\delta(\underset{\sim}{k}+\underset{\sim}{k}'+\underset{\sim}{k}'')e^{i\underset{\sim}{k}'\cdot\underset{\sim}{r}'+i\underset{\sim}{k}''\cdot\underset{\sim}{r}''-is'\Omega(\underset{\sim}{k}')\tau'-is''\Omega(\underset{\sim}{k}'')\tau''}$$

$$e^{-i\left[s\Omega(\underset{\sim}{k})+s'\Omega(\underset{\sim}{k}')+s''\Omega(\underset{\sim}{k}'')\right]t} \qquad . \tag{4.15}$$

The Fourier component $\hat{U}^{s}_{\underset{\sim}{k}}$ can be found in terms of $\hat{G}^{s}_{\underset{\sim}{k}}$ for whatever Lagrangian variable $u(\underset{\sim}{x},t)$ represents. For example, if $w(\underset{\sim}{x},t)$ represents vertical velocity $u_3(\underset{\sim}{x},t)$, then $\hat{W}^{s}_{\underset{\sim}{k}} = is\Omega(\underset{\sim}{k})\left(G^{s}_{\underset{\sim}{k}}\right)_3$ since $u_3(\underset{\sim}{x},t) = \frac{\partial}{\partial t}\zeta_3(\underset{\sim}{x},t)$. Equation (4.15) is the discreet equivalent to (2.6), so that the discreet bispectrum of u, v, w is given by

$$B_{uvw}(\underset{\sim}{k}',\omega',\underset{\sim}{k}'',\omega'';t)$$

$$= \hat{U}^{s}_{\underset{\sim}{k}}\hat{V}^{s'}_{\underset{\sim}{k}'}\hat{W}^{s''}_{\underset{\sim}{k}''}\;\langle a^{s}(\underset{\sim}{k},t)a^{s'}(\underset{\sim}{k}',t+\tau')a^{s''}(\underset{\sim}{k}'',t+\tau'')\rangle$$

$$\delta(\underset{\sim}{k}+\underset{\sim}{k}'+\underset{\sim}{k}'')\;e^{-i\left[s\Omega(\underset{\sim}{k})+s'\Omega(\underset{\sim}{k}')+s''\Omega(\underset{\sim}{k}'')\right]t} \tag{4.16}$$

with

$$\omega' \equiv -s'\Omega(\underset{\sim}{k}'), \qquad\qquad \omega'' \equiv -s''\Omega(\underset{\sim}{k}'') \quad .$$

At this point, no assumption of stationarity has been made. Introducing (4.9) into the ensemble average triple product of the wave amplitudes in (4.16) yields

$$\langle a^{s}(\underset{\sim}{k},t)a^{s'}(\underset{\sim}{k}',t+\tau')a^{s''}(\underset{\sim}{k}'',t+\tau'')\rangle =$$

$$\varepsilon^3 \langle a_1^s(\underset{\sim}{k},0)a_1^{s'}(\underset{\sim}{k}',0)a_1^{s''}(\underset{\sim}{k}'',0)\rangle$$

$$+ \varepsilon^4 \langle a_2^s(\underset{\sim}{k},t)a_1^{s'}(\underset{\sim}{k}',0)a_1^{s''}(\underset{\sim}{k}'',0)\rangle$$

$$+ \varepsilon^4 \langle a_1^s(\underset{\sim}{k},0)a_2^{s'}(\underset{\sim}{k}',t+\tau')a_1^{s''}(\underset{\sim}{k}'',0)\rangle$$

$$+ \varepsilon^4 \langle a_1^s(\underset{\sim}{k},0)a_1^{s'}(\underset{\sim}{k}',0)a_2^{s''}(\underset{\sim}{k}'',t+\tau'')\rangle$$

$$+ \ldots \quad . \tag{4.17}$$

The first term is zero because the first order field is Gaussian. The next set of terms will not be zero however, as the second order amplitudes depend on products of first order amplitudes so that these terms are averages of fourth order products of the first order field.

To this point the development of theoretical bispectra has followed that in Hasselmann *et al.* (1963) (with the exception of the Lagrangian representation). However, Hasselmann a priori assumed that the third order statistics that arise due to nonlinear interaction of an initially Gaussian field will be stationary. This is in general not true. Bispectra of strongly interacting phenomenon such as turbulence, arising from an initially Gaussian state, (This does not preclude stationarity for strongly interacting phenomena: It does indicate that the assumptions of strict stationarity and "quasi-Gaussianity" are incompatible for such motions.) will not be stationary. Clearly

the degree of stationarity will depend upon the weakness of the non-linearity. For ocean waves, which do not admit quadratic resonant interactions, strict stationarity is not possible.

For internal waves, which do admit quadratic resonant interactions, the statistics asymptotically reach stationary values as it can be shown that in the limit of infinitely weak interactions (4.17) becomes

$$\lim_{t \to \infty} \; \langle a^{s}(\underset{\sim}{k},t)a^{s'}(\underset{\sim}{k}',t+\tau')a^{s''}(\underset{\sim}{k}'',t+\tau'')\rangle$$

$$= \frac{-i\pi\Omega(\underset{\sim}{k})\Omega(\underset{\sim}{k}')\Omega(\underset{\sim}{k}'')}{8\,\rho_o \gamma_{\underset{\sim}{k}}\gamma_{\underset{\sim}{k}'}\gamma_{\underset{\sim}{k}''}} D_{-\underset{\sim}{k}'-\underset{\sim}{k}''-\underset{\sim}{k}}^{-s'-s''-s} \delta(\underset{\sim}{k}+\underset{\sim}{k}'+\underset{\sim}{k}'')$$

$$\delta\left(s\Omega(\underset{\sim}{k})+s'\Omega(\underset{\sim}{k}')+s''\Omega(\underset{\sim}{k}'')\right)$$

$$\left[sA(\underset{\sim}{k}')A(\underset{\sim}{k}'')+s''A(\underset{\sim}{k}')A(\underset{\sim}{k})+s'A(\underset{\sim}{k}'')A(\underset{\sim}{k})\right] \qquad (4.18)$$

where $\hat{A}(\underset{\sim}{k}) = \hat{E}(\underset{\sim}{k})/\Omega(\underset{\sim}{k})$ is the wave action spectrum. Substituting (4.18) into (4.16) gives the weak interaction formula for the stationary bispectrum of anv Lagrangian variables u, v, w

$$B_{uvw}(\underset{\sim}{k}',\omega',\underset{\sim}{k}'',\omega'') =$$

$$- \frac{i\pi\Omega(\underset{\sim}{k})\Omega(\underset{\sim}{k}')\Omega(\underset{\sim}{k}'')}{8\rho_o \gamma_{\underset{\sim}{k}}\gamma_{\underset{\sim}{k}'}\gamma_{\underset{\sim}{k}''}} D_{-\underset{\sim}{k}'-\underset{\sim}{k}''-\underset{\sim}{k}}^{-s'-s''-s} \hat{U}_{\underset{\sim}{k}}^{s}\hat{V}_{\underset{\sim}{k}'}^{s'}\hat{W}_{\underset{\sim}{k}''}^{s''}$$

$$\delta(\underline{k}+\underline{k}'+\underline{k}'')\delta\left(s\Omega(\underline{k})+s'\Omega(\underline{k}')+s''\Omega(\underline{k}'')\right)$$

$$\left[s\hat{A}(\underline{k}')\hat{A}(\underline{k}'')+s''\hat{A}(\underline{k}')\hat{A}(\underline{k})+s'\hat{A}(\underline{k}'')\hat{A}(\underline{k})\right] \tag{4.19}$$

Thus the bispectrum of <uvw> is given by an interaction coefficient times second order products of the action density spectrum. The advantage of the Lagrangian description and the use of the action density is that the coefficient is the same for each of the three spectral product terms. Eulerian bispectra can be generated from the Eulerian equations of motion, but with considerably more effort and with less appealing results.

Hasselmann (1966) showed that further consideration of the amplitude expansion (4.9) leads to an expression for the time rate of change of the action spectrum, i.e.,

$$\frac{\partial\hat{A}(\underline{k})}{\partial t} = \sum_{\underline{k}'}\sum_{\underline{k}''}\sum_{s'}\sum_{s''} \frac{\pi s\Omega(\underline{k})\Omega(\underline{k}')\Omega(\underline{k}'')}{8\rho_o\gamma_{\underline{k}}\gamma_{\underline{k}'}\gamma_{\underline{k}''}} \left|D^{-s'-s''-s}_{-\underline{k}'-\underline{k}''-\underline{k}}\right|^2$$

$$\delta(\underline{k}+\underline{k}'+\underline{k}'')\delta(s\Omega(\underline{k})+s'\Omega(\underline{k}')+s''\Omega(\underline{k}''))$$

$$\left[s\hat{A}(\underline{k}')\hat{A}(\underline{k}'')+s''\hat{A}(\underline{k}')\hat{A}(\underline{k})+s'\hat{A}(\underline{k}'')\hat{A}(\underline{k})\right] . \tag{4.20}$$

Note the remarkable similarity between (4.20) and (4.19). The only significant difference is the interaction coefficient. Where (4.20)

has $\left|D^{-s'-s''-s}_{-\underline{k}'-\underline{k}''-\underline{k}}\right|^2$, (4.19) has $D^{-s'-s''-s}_{-\underline{k}'-\underline{k}''-\underline{k}}\ U^{s}_{\underline{k}}V^{s'}_{\underline{k}'}W^{s''}_{\underline{k}''}$. This near equivalence will be exploited later.

The rate of change of the action spectrum is simply related to the rate of change of the energy spectrum by

$$\frac{\partial \hat{E}(\underline{k})}{\partial t} = \Omega(\underline{k})\ \frac{\partial \hat{A}(\underline{k})}{\partial t} \tag{4.21}$$

so that the integrand of (4.20) times $\Omega(\underline{k})$ gives the rate of change of the energy spectrum at wave number $\underline{k}$ and frequency $\Omega(\underline{k})$ by interaction of two waves $\underline{k}'$, $s'\Omega(\underline{k})$ and $\underline{k}''$, $s''\Omega(\underline{k}'')$. Only those waves whose wave numbers and frequencies sum to zero make any contribution. This is precisely the definition of the bispectrum of power of section 3. Hence (or it can be shown rigorously) the bispectrum of power arising from weak interaction is given by

$$\begin{aligned}\mathrm{Re}\ \left[\hat{S}(\underline{k}',s'\Omega(\underline{k}'),\underline{k}'',s''\Omega(\underline{k}''))\right] = \\ \Omega(\underline{k})\ \frac{\pi s\Omega(\underline{k})\Omega(\underline{k}')\Omega(\underline{k}'')}{8\rho_o\gamma_{\underline{k}}\gamma_{\underline{k}'}\gamma_{\underline{k}''}}\ \left|D^{-s'-s''-s}_{-\underline{k}'-\underline{k}''-\underline{k}}\right|^2\ \delta(\underline{k}+\underline{k}'+\underline{k}'') \\ \delta(s\Omega(\underline{k})+s'\Omega(\underline{k}')+s''\Omega(\underline{k}'')) \\ \left[s\hat{A}(\underline{k}')\hat{A}(\underline{k}'')+s''\hat{A}(\underline{k}')\hat{A}(\underline{k})+s'\hat{A}(\underline{k}'')\hat{A}(\underline{k})\right]\ .\end{aligned} \tag{4.22}$$

Given some wave action spectrum, such as provided by the GM models, the bispectrum of power or any Lagrangian bispectrum can be numerically evaluated using (4.22) or (4.19) respectively. Comparison of the predicted bispectra to the observed bispectra is a test of the adequacy of weak interaction theory. Production of such theoretical bispectra is the objective of the next section.

V. Evaluating Theoretical Bispectra

The previous section gave the expression for bispectra of a weakly interacting random internal wave field. In this section the bispectrum of power and the auto-bispectrum of Lagrangian vertical displacements will be evaluated numerically for the GM model of the internal wave spectrum.

5.1 Frequency Bispectra

5.1.1 The Bispectrum of Power

The full bispectral density $\hat{S}(\underline{k}',s',\Omega(\underline{k}'),\underline{k}'',s''\Omega(\underline{k}''))$ is given by (4.22). As the frequency $\omega' = s'\Omega(\underline{k}')$ is a function of the wave number alone, the bispectrum of power is a function of the six components of $\underline{k}'$, $\underline{k}''$. Due to the delta function in frequency, only those waves which satisfy the frequency resonance conditions have a non-zero density. Thus, the bispectral density can be reduced to a function of five arguments, without loss of generality.

At this point the theoretical bispectrum is fairly simple to evaluate, as only one triad contributes to each point in the 5-dimensional bispectral density. However the task of observing such a bispectrum, which would require records in time and two space dimensions, or all three space dimensions, seems quite hopeless. If a comparison between theory and observation is to be made, the number of arguments must be reduced. As bispectra of two arguments in one space dimension or time seems to be the most readily observed, only frequency or vertical wave number bispectra will be evaluated.

The reduction of the theoretical bispectra from five arguments to two requires three integrations, a considerable task. As the GM model spectra are horizontally isotropic, an integration in horizontal direction is trivial, leaving only two integrations to be performed. The procedure is as follows: the two arguments of the bispectral density are chosen, two values of the integration domain are selected, the integrand given by (4.24) is evaluated using the GM model for the

action spectrum, another point in the integration domain is selected, and so on. Integration over all the computed values gives the bispectrum for that particular choice of the two arguments. The procedure is repeated for another choice of the two arguments until the entire bispectrum is obtained.

Figure 2 shows the result of such a computation - the frequency bispectrum of power. Inspection of (3.15) shows that the bispectrum of power of two arguments can be displayed in a quarter plane with

$$\hat{S}(\omega',\omega'') = \hat{S}(\omega'',\omega') \tag{5.1}$$

(Section 2.3). The interactions of octant 1 can be thought of as a "sum" reaction, i.e.

$$\Omega(\underline{k}') + \Omega(\underline{k}'') = \Omega(\underline{k}) \tag{5.2}$$

where $\Omega(\underline{k})$ interacts with two smaller frequencies $\Omega(\underline{k}')$ and $\Omega(\underline{k}'')$. Octant 8 as a "difference" reaction

$$\Omega(\underline{k}') - \Omega(\underline{k}'') = \Omega(\underline{k}) \tag{5.3}$$

where $\Omega(\underline{k})$ interacts with one larger frequency and a larger or smaller frequency. A wave number bispectrum has the same magnitude relations for these octants, however, the designation as sum and difference reactions are not as useful because there is no convention for $\underline{k} \geq 0$ like that for $\Omega(\underline{k}) \geq 0$.

In Figure 2 the largest bispectral values are found at the very lowest frequencies. The region is displayed in greater detail in Figure 3. The large negative value in the upper left-hand region shows that energy is being transferred from the frequency band $\omega = 2f - 3f$ by interaction with nearly half frequency waves $\omega' \simeq \omega'' = 1f - 1.5f$. As shown by the large positive values in the center of the plot, energy is also being gained at $\omega = 1.0f - 1.5f$ by interactions with nearly twice that frequency (ω') and nearly the same frequency (ω''). The open regions have zero bispectral values because at least one of the frequencies is less than the local inertial frequency, and no interactions are possible.

These large transfers out of the frequency band $\omega = 2.0f - 3.0f$ and into $\omega = 1.0f - 1.5f$, the inertial peak, have been identified as a parametric subharmonic instability by MB. The waves involved in such a triad interaction are shown schematically in Figure 4(a). MB also found that this interaction gave the strongest rate of change for the energy spectrum, rapidly transferring energy down scale and into near inertial waves. McComas (1977) has suggested that this is the most effective mechanism for delivering energy to the small scale dissipation region.

It should be noted that agreement between these bispectral calculations and the results of MB is to be expected, as the bispectrum of power is merely another way of displaying the same quantity - the time rate of change of energy. The primary objective is to provide a test of the theoretical predictions and proposed mechanisms. Thus it is important to determine the bispectral signature of these mechanisms and the feasibility of an appropriate observational program.

The frequency bispectrum of power shows a ridge of high values along the ω' axis for large values of ω'. This corresponds to the induced diffusion interaction of MB, an interaction involving two nearly identical waves ω, ω' interacting with a much lower frequency and much smaller wave number wave ω'' as in Fig. 4(b). In such an interaction, wave action is diffused in wave number space. Because of the particular shape of the internal wave spectrum represented by the GM models, the rate of change of action (or energy) of a given wave component ω is much smaller than if the spectrum had only a slightly different shape (McComas 1977). This indicates some kind of interaction equilibrium in the internal wave field. Such a balance is clearly shown in Fig. 2. The large negative value in octant 1 has a counterpart positive value in octant 8. Figure 5 gives sections of the frequency bispectrum for various values of ω. The dominance of the induced diffusion interaction for the time rate of change of the energy spectrum at these frequencies is dramatic. It is also quite apparent that the total result of the interaction is much smaller because the contributions from the two octants nearly cancel. Any small change in the spectrum that disturbs that near balance would result in a large increase in the transfer rate.

The positive and negative transfers in the two octants indicates a strong energy throughput to lower frequency. Recall that octant 1, a "sum" interaction, involves interactions of ω with two lower frequencies ω' and ω''. The negative value in that region implies that energy is being lost from ω to the nearby ω' and much smaller

ω'' . In the "difference" interaction octant, energy is being gained from interaction with slightly larger ω' and much smaller ω'' . Since the induced diffusion interaction transfers energy primarily between the two high frequency components (MB), this result indicates a gain of energy at frequency ω from slightly higher frequencies and a nearly equal loss to lower frequencies. That amounts to a flow through ω to lower frequencies. Its magnitude ranges from approximately 2.5×10^{-8} J/m^3s at $\omega = 20f$ to 2.5×10^{-7} J/m^3s at $\omega = 60f$, the same order as Müller and Olbers' (1975) estimate of the internal wave dissipation. Furthermore, that throughput is nearly non-divergent so that the energy in ω remains fairly constant. This result is new, since the procedure of MB was unable to resolve the direction of the energy flow, and it nicely supports McComas' (1977) assumption of a flux by this mechanism to a low frequency, high wave number dissipation. As noted by MB such a flux implies a gain of energy at the low frequency component, and that gain is reflected in Fig. 2 for ω small and $\omega' \gtrsim -\omega''$ large.

The strong interactions identified by MB show highly individualistic signatures in the frequency bispectrum of power, and the foregoing types of arguments about energy flux and balances within the spectrum show that this bispectrum is a potentially powerful indicator of the nonlinear interaction and resulting energy transfer within the wave spectrum. Obtaining such a bispectrum from observation would be extremely valuable.

Unfortunately, bispectra are quite difficult to measure with any reasonable confidence unless the nonlinearities are quite

strong. For example, Lii *et al* (1976) required nearly 10,000 estimated degrees of freedon, e.d.o.f., to obtain minimal confidence for wave number bispectra of wind tunnel turbulence, presumably a strongly nonlinear phenomenon! As the features of interest for the internal wave bispectrum require frequency resolution of the order of the inertial frequency, i.e., a record length of at least one day, Lii's results suggest that an experiment of 10^4 days is required if the internal wave nonlinearities are as strong as wind tunnel turbulence! Such an experiment is of course impossible, but even if it were done, such strong nonlinearities would violate the basic assumption of statistical stationarity.

The required number of e.d.o.f. for the 95% confidence level is (Haubrich 1965)

$$\nu = 4 \left[C^2_{\hat{S}}(\omega', \omega''; \Delta\omega', \Delta\omega'') \right]^{-1} \tag{5.4}$$

where $C_{\hat{S}}$ is the bicoherence defined in (2.26) for the bispectrum of power. An order of magnitude estimate of the bicoherence can be obtained from

$$\begin{aligned} C^2_{\hat{S}}(\omega', \omega''; \Delta\omega', \Delta\omega'') &\approx \frac{\left| \hat{S}(\omega', \omega'') \Delta\omega' \Delta\omega'' \right|^2}{E(\omega) E(\omega') S(\omega'')} \\ &\approx \left(\frac{T_{INT}}{T_N} \right)^{-2} \frac{E(\omega)}{E(\omega')} \frac{S}{S(\omega'')} \end{aligned} \tag{5.5}$$

where $S \approx N^2$ is the total shear, $S(\omega)$ is the shear in frequency band $\Delta\omega$, T_{INT} is the e-folding time scale for the energy in frequency band $\Delta\omega$ by interaction with frequency bands $\Delta\omega'$ and $\Delta\omega''$, and T_N is the Brunt-Väisälä period. Taking the strong induced diffusion interaction with $\omega' = 40f$, $\omega'' = 1.5f$, $\Delta\omega = \Delta\omega' = \Delta\omega'' = 1f$, it results within an order of magnitude that

$$O(\nu) \approx \left(\frac{T_{INT}}{T_N}\right)^2 \tag{5.6}$$

such that for a ten day experiment, $T_{INT} \approx 3\ T_N$ i.e. the e-folding time of the energy in frequency band $\Delta\omega$ due to interaction only with frequency bands $\Delta\omega'$ and $\Delta\omega''$ has to be the order of the Brunt-Väisälä period. That implies extremely strong nonlinearities!

The squared bicoherence, roughly the inverse of the number of degrees of freedom required to measure $\hat{S}(\omega',\omega'')$, for a band width of 1f has been computed using (5.5) and is given in Fig. 6. Even the strong induced diffusion reaction would require one thousand e.d.o.f.! The strong nonlinearities required to produce an observable bispectrum with only a few degrees of freedom would probably preclude the required stationarity. Although the frequency bispectrum of power would be extremely interesting and valuable, its determination from observation, at least open ocean observations, is impossible.

5.1.2 The Frequency Auto-bispectrum of Vertical Displacements

The very low level of the frequency bispectrum of power may be peculiar to that particular bispectrum. Perhaps some other bispectrum, which clearly demonstrates the strong interaction mechanisms, would have an observable level. Neshyba and Sobey (1975) reported highly significant levels for frequency bispectra of vertical displacements, and they suggested that the interaction trapping mechanism of Phillips (1968) was responsible. This mechanism, called elastic scattering by MB, reflects up-going waves into down-going waves, and vice versa and forces the spectrum into vertical symmetry in a time the order of one period and in a vertical distance the order of one wavelength (McComas 1977). A typical elastic scattering triad is shown in Figure 4(c). As this bispectrum promises to be a good measure of the elastic scattering mechanisms and observable levels have already been reported, this section evaluates the frequency auto-bispectrum of Lagrangian vertical displacements.

Any bispectrum arising from weak nonlinear interactions may be evaluated from (4.19). The auto-bispectrum of vertical displacements is easier than most as the Fourier component for vertical displacements has been chosen as unity, i.e.

$$\left(\hat{\xi}_3\right)^{s}_{\underline{k}} \equiv 1 \qquad (5.7)$$

so that

$$B_{\xi_3\xi_3\xi_3}\left(\underline{k}', s'\Omega(\underline{k}'), \underline{k}'', s''\Omega(\underline{k}'')\right) = \frac{RE\left[\hat{S}\left(\underline{k}', s'\Omega(\underline{k}'), \underline{k}'', s''\Omega(\underline{k}'')\right)\right]}{-i\Omega(\underline{k})D^{s's''s}_{\underline{k}'\underline{k}''\underline{k}}} \tag{5.8}$$

The similarity of form between the auto-bispectrum of Lagrangian vertical displacements $B_{\xi_3\xi_3\xi_3}$ and the bispectrum of power $\hat{S}$ not only saves considerable effort in their evaluation, but also suggests that the much more readily observed vertical displacement bispectrum might be a good indicator of the strong energy transfer mechanisms.

A detailed inspection of the form of $B_{\xi_3\xi_3\xi_3}$ reveals that this bispectrum is zero if the spectrum is vertically symmetric. This results because for every "up" triad there is a "down" triad that precisely cancels the "up" triad's bispectral value. In a perfectly vertically symmetric spectrum, this bispectrum is everywhere zero. A significant non-zero level is therefore an indication of vertical asymmetry in the spectrum.

McComas (1977) has shown that the elastic scattering mechanism eliminates a 10% perturbation to the vertical symmetry of the GM spectrum on the order of one period (at the shorter scales) - a very rapid relaxation. As a vertically symmetric spectrum has no vertical displacement bispectrum, some asymmetry is required and the

same 10% perturbation, namely 10% more energy in down-going waves and 10% less in up-going waves than specified by the GM model, is used. Figure 7 shows the resulting frequency auto-bispectrum. It is everywhere positive because the down-going waves have more energy. Its shape is considerably more simple than the bispectrum of power. The predominant feature is the ridge of high values for ω'' small. This corresponds to the strong interaction of the elastic scattering mechanism. (Although the induced diffusion interaction has the same frequency characteristics, it does not appear in this bispectrum because of the near cancellation by "up" and "down" triads.) A similar ridge was observed by Neshyba and Sobey (1975). However, the level of their bispectrum was five orders of magnitude greater than here!

Again one must ask if the predicted level is observable. Figure 8 shows that this bispectral level is hopelessly low, in apparent contradiction to Neshyba and Sobey's results. The level is so low that it is not likely that the discrepancy can be due to differences in the basic spectrum, even though Neshyba and Sobey's arctic internal wave spectrum may have been substantially different from the deep, open ocean spectrum represented by the GM model. Rather it is due to the fact that these are not bispectra of the same quantities! The theoretical prediction is for Lagrangian vertical displacements, but the observed bispectrum is taken from records of undulating step-like layers, and hence is neither an Eulerian or Lagrangian measurement. It is approximately Eulerian in horizontal

position, but neither Lagrangian nor Eulerian in vertical position. Their vertical displacement $\xi_3(\underline{x},z;t)$ is related to the true Lagrangian vertical displacement $\xi_3(\underline{X},Z;t)$ by (2.19) so that

$$
\begin{aligned}
&\langle \xi_3(\underline{x},z;t)\xi_3(\underline{x},z;t)\xi_3(\underline{x},z;t)\rangle = \\
&\quad \langle \xi_3(\underline{X},Z;t)\xi_3(\underline{X},Z;t)\xi_3(\underline{X},Z;t)\rangle \\
&\quad - \langle \xi_i(\underline{X},Z;t)\frac{\partial \xi_3(\underline{X},Z;t)}{\partial x_i}\xi_3(\underline{X},Z;t)\xi_3(\underline{X},Z;t)\rangle \\
&\quad - \langle \xi_3(\underline{X},Z;t)\xi_i(\underline{X},Z;t)\frac{\partial \xi_3(\underline{X},Z;t)}{\partial x_i}\xi_3(\underline{X},Z;t)\rangle \\
&\quad - \langle \xi_3(\underline{X},Z;t)\xi_3(\underline{X},Z;t)\xi_i(\underline{X},Z;t)\frac{\partial \xi_3(\underline{X},Z;t)}{\partial x_i}\rangle \\
&\quad + 0\langle \xi^5\rangle \quad .
\end{aligned}
\tag{5.9}
$$

$$i = 1,2$$

As pointed out in section 2.4, fourth order terms are non-zero even when the Lagrangian field is Gaussian.

The fourth order correlations in (5.9) can be considered as a third order correlation between ξ_3,ξ_3, and $\xi_i \frac{\partial \xi_3}{\partial x_i}$, a "false" signal which arises because horizontal gradients of ξ_3 are

advected past the horizontally fixed sensor. A large value of $\xi_i \frac{\partial \xi_3}{\partial x_i}$ might be expected from high frequency components of ξ_3 (large horizontal wave number) and low frequency ξ_i (large displacements). Such a signal would automatically be correlated with one high frequency ξ_3 and one low frequency ξ_3, producing a high bispectral value precisely where Neshyba and Sobey found their strongest results. In the opinion of the author, their findings are a result of the kinematics of the sampling procedure - not the dynamics of the nonlinear internal wave field.

5.2 Vertical Wave Number Bispectra

The previous sections found two frequency bispectra which clearly reflected the energy transfer mechanisms of MB and would have provided observational tests of these mechanisms had their level been sufficient to be reliably determined by an observational experiment of reasonable length. MB found that small scale waves have much shorter e-folding times than large scale energy containing waves. Although vertical wave number bispectra will combine all frequencies in one wave number band, just as the frequency bispectra included all wave number scales in one frequency band, the stronger nonlinearity of these small scale waves might produce an observable bispectral level. For example, consider the required e.d.o.f. for the induced diffusion interaction where $k_3' \approx k_3$ is large and k_3'' is small. From (5.5)

$$\nu \approx \left(\frac{T_{INT}}{T_N}\right)^2 \frac{E(k_3)}{E(k_3')} \frac{S}{S(k_3'')} \tag{5.10}$$

much as before. However, because k_3'' is a small wave number it contributes only a small part of the shear - perhaps only one hundredth of the total shear - so that for $\nu = 10$, $T_{INT} = 30\ T_N$: a ten times slower rate than required for observable frequency bispectra.

If the statistics are stationary, an ensemble may be produced from vertical profiles taken at different locations and times (assuming homogeneity and stationarity), or from samples (at least a correlation time apart) of time histories of vertically separated instruments.

With the faster interaction times found by MB at the smaller scales, the vertical wave number bispectrum of power might have observable levels.

Figure 9 shows the theoretical vertical wave number bispectrum of power. Because of the large range of scales in the internal wave field, this bispectrum is best displayed on logarithmic axes. In order to utilize the scaling of the resonance conditions that make the numerical computation of this bispectrum possible, the vertical axes is not k_3'' but $k_3''/|k_3'|$. The unique quarter plane of the bispectrum of power now lies in $-1 \leq k_3''/|k_3'| \leq 1$. On such an axis system, the third wave number is not constant along a line of -1 slope, as on the linear axes (Figure 1), but, as before, may be determined from $k_3 + k_3' + k_3'' = 0$. Finally, each bispectral value has been multiplied by $k_3'k_3''$ so that the plotted bispectral value times the area $(20)^2$ $d\ell og_{10}|k_3''k_3'| \; d\ell og_{10}|k_3'|$ is the contribution to $\frac{\partial E(k_3)}{\partial t} dk_3$ for that area. Triads with waves of similar scale are found at the top of the plot ($k_3''/|k_3'| \sim 1$), triads with k_3' large and k_3'' small are in the center, and triads with k_3' nearly $-k_3''$ and with k_3 much smaller are at the bottom.

Note that in this representation the bispectrum is remarkably constant, indicating that each logarithmic area transfers roughly comparable amounts of energy. This of course implies that the large vertical wave numbers are more nonlinear since their energy content is much lower.

At least half the area of the plot corresponds to interaction of triads with widely separated scales ($k_3 \sim k_3' \sim 10|k_3''|$) - the parametric subharmonic instability, and the induced diffusion interactions (Figure 4a and b). MB showed that $|k_3''/k_3'| = \omega''/\omega'$ for the induced diffusion triad, and as frequencies are bounded, the interaction is found only in a restricted range of $k_3''/|k_3'|$. The largest positive values in the upper region and the largest negative values in the lower region (both for $k_3''/|k_3'| \leq 0.1$) are a result of the induced diffusion interaction. This is the opposite set of signs found for the frequency bispectrum (a throughput to smaller frequency), and using similar arguments, implies a throughput to larger vertical wave numbers. The throughput is approximately independent of vertical wave number and is of the order of 1×10^{-10} Joules/m^3s for a bandwidth of 1/20 of a decade. This is a rather small throughput compared to the Müller-Olbers' dissipation estimate of 2 to 8×10^{-7} J/m^3s and seems to reflect the near balance of the GM76 model with respect to this mechanism.

For smaller values of $k_3''/|k_3'|$, there is a positive transfer due to the parametric subharmonic instability into the large vertical wave numbers k_3 and k_3' out of k''. These transfers are smaller than the throughput by the induced diffusion triad, but this interaction is capable of transferring energy directly from the small wave number component to the large wave number components and is not the cascading process of the induced diffusion interaction. Further, the plots

arbitrarily stop at $|k_3''/k_3'| \leq 0.01$, yet for smaller values this interaction is still contributing, and one may conclude that this mechanism is an important energy transfer mechanism. The loss from the small wave number component of both the induced diffusion interaction and the parametric instability (the major component) is found at $k_3''/|k_3'| \sim -1.0$.

Finally we note that there is a ridge of interactions at $k_3''/|k_3'| = -0.5$ corresponding to the elastic scattering interaction. It does not stand out as a large transfer rate but only a change in sign of the transfer. The transfer out of the low frequency, double wave number component is found at $k_3''/|k_3'| \sim 1.0$ and is much smaller as is appropriate for the much lower frequency wave (MB).

Figure (10) shows the bicoherence of the vertical wave number bispectrum of power. Clearly, this bispectrum is not readily observed either. The strongest bispectral value is associated with the induced diffusion interaction, requiring more than 10^3 e.d.o.f. Apparently the frequency masking within each waveband is more serious than implied by the simple order of magnitude estimate at the beginning of this section.

VI. Summary and Remarks on an Observational Program

Bispectra that directly indicate the energy transfer among components of the internal wave field may be derived from the equations of motion. The same bispectra can also be evaluated numerically from weak interaction theory, providing a possible test of that theory and its prediction of strong interaction mechanisms. However, the computations indicate that the level of these bispectra require extensive amounts of data to achieve statistical significance.

The level of the frequency bispectrum and the long time required to obtain a degree of freedom (1 day) precludes the feasibility of any successful observational program. Although the statistically significant result of Neshyba and Sobey (1975) seems to be explained as a result of nonhomogeneities of that particular wave field and the way in which it was sampled, a check on the "unobservable level" prediction for the frequency bispectrum of vertical displacement from a clean Lagrangian measurement in the typical open ocean conditions is recommended. Hopefully, this will be available shortly (M. Briscoe, personal communication).

The level of the vertical wave number bispectrum of power requires many degrees of freedom for statistical significance, and as one must wait at least a correlation time before the next sample, is probably no easier to observe than the frequency bispectrum. In view of the probable expense and the limited chance for success, a program to obtain this bispectrum as a limited check of weak interaction theory is not recommended.

In final summary, it seems that bispectra are too insensitive a test for observing the nonlinear dynamics of the internal wave field.

and horizontal gradients of velocity and buoyancy. The last record presents the greatest difficulty, however a vertical profile of the horizontal gradient of the horizontal and vertical velocities will also be quite difficult. No presently available instrument package is capable of acquiring these records. Some type of acoustic instrument with a fairly long acoustic path to average out fine structure gradients from the internal wave gradients is probably required. However, even the strongest interactions require over 10^3 samples, and even if the result varied greatly from the theoretical prediction, that would not be very surprising as the assumption of "weak" nonlinearity is clearly violated at these small scales. In view of the probable expense and the limited chance of success (any nonstationarity being fatal), a program to obtain this bispectrum as a limited check of weak interaction theory is not recommended.

In final summary, bispectra are too insensitive a technique for observing the nonlinear dynamics of the internal wave field. Perhaps more limited and indirect measures of the energy dynamics can be gathered, such as correlations in energy content between portions of the spectrum or changes in spectral shape during times of energy input or dissipation. The interpretation of such measures will probably rely heavily on the theoretical mechanisms and, therefore, may not be clean tests of the accuracy of these theoretical ideas. It is unfortunate that the direct bispectral test is not possible.

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Figure 1. Representation of display plane for bispectra. If the axes are linear, the third frequency is constant along a line of -1 slope, represented here for a bandwidth $\Delta\omega$. The dashed lines passing through the origin indicate the "fold lines" about which certain bispectra are symmetric.

Figure 2. (a) The frequency bispectrum of power contoured in powers of ten in m^2/s. Dashed contours indicate negative values. Thin solid lines represent boundaries outside of which the bispectral level is zero because one of the frequencies is smaller than f or larger than N. The dashed pattern indicates the symmetry fold lines. (b) A three dimension representation of the top half of (a) with symmetry about the fold line, looking toward the origin with viewing angle I. The vertical axis is logarithmic. (c) A three dimensional representation of the bottom half of (a) with symmetry about the fold line, looking toward the origin with viewing angle II.

Figure 3. Detail on the region near the origin of Figure 2 showing the transfers into the inertial band $1 \leq \omega/f \leq 2$ out of the band $2 \leq \omega/f \leq 4$. The representation is rotated 45^{o} from the previous figure so that lines of constant ω are horizontal. The numbers at the corners represent points in (ω', ω'') space.

Figure 4. Schematic representation of the wave number and frequency characteristics of resonant triads in the (a) parametric subharmonic instability (b) induced diffusion and (c) elastic scattering interactions.

Figure 5. Sections of the frequency bispectrum of power along (a) AA' for $\omega/f = 20$, (b) BB' for $\omega/f = 40$, and (c) CC' for $\omega/f = 60$.

Figure 6. Contour plot of the squared bicoherence for the frequency bispectrum of power. Only the upper octant is shown.

Figure 7. Contour plot of the frequency auto-bispectrum of Lagrangian vertical displacements in m^3s^2. Only the unique octant is shown.

Figure 8. Contour plot of the squared bicoherence of the frequency auto-bispectrum of Lagrangian vertical displacements.

Figure 9. Contour plot of the vertical wave number bispectrum of power times $k_3'k_3''$ in m^2/s^2. The actual volume under the plot gives the total contribution to $\frac{\partial E(k_3)}{dt} dk_3$.

Figure 10. Contour plot of the squared bicoherence of the vertical wave number bispectrum of power.

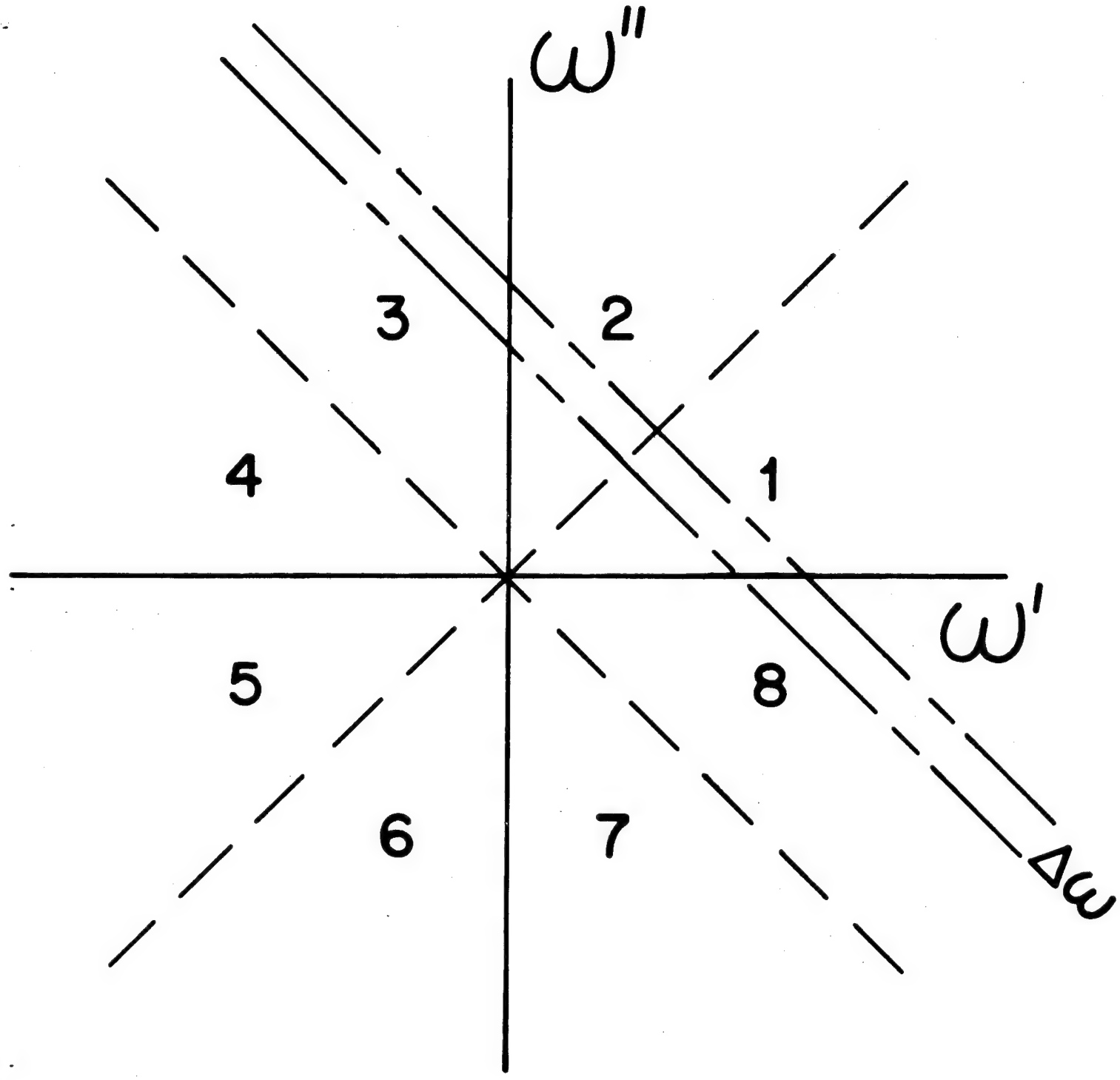

Figure 1

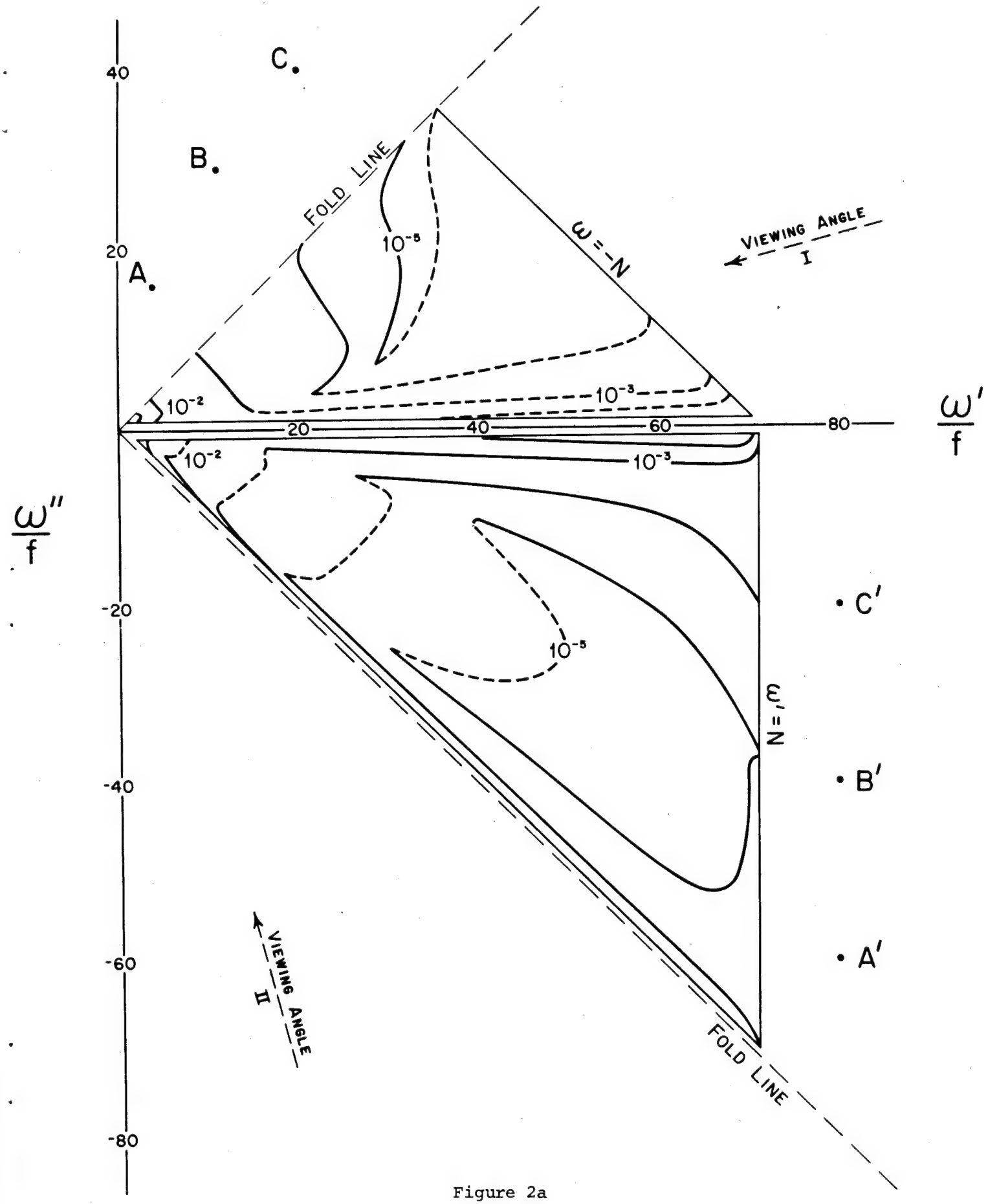
C.
B.
A.
FOLD LINE
10^{-5}
$\omega = -N$
VIEWING ANGLE
I
10^{-2}
10^{-3}
40
20
-20
-40
-60
-80
20
40
60
80
$\frac{\omega'}{f}$
$\frac{\omega''}{f}$
10^{-2}
10^{-3}
10^{-5}
$\omega' = N$
C'
B'
A'
VIEWING ANGLE
II
FOLD LINE

Figure 2a

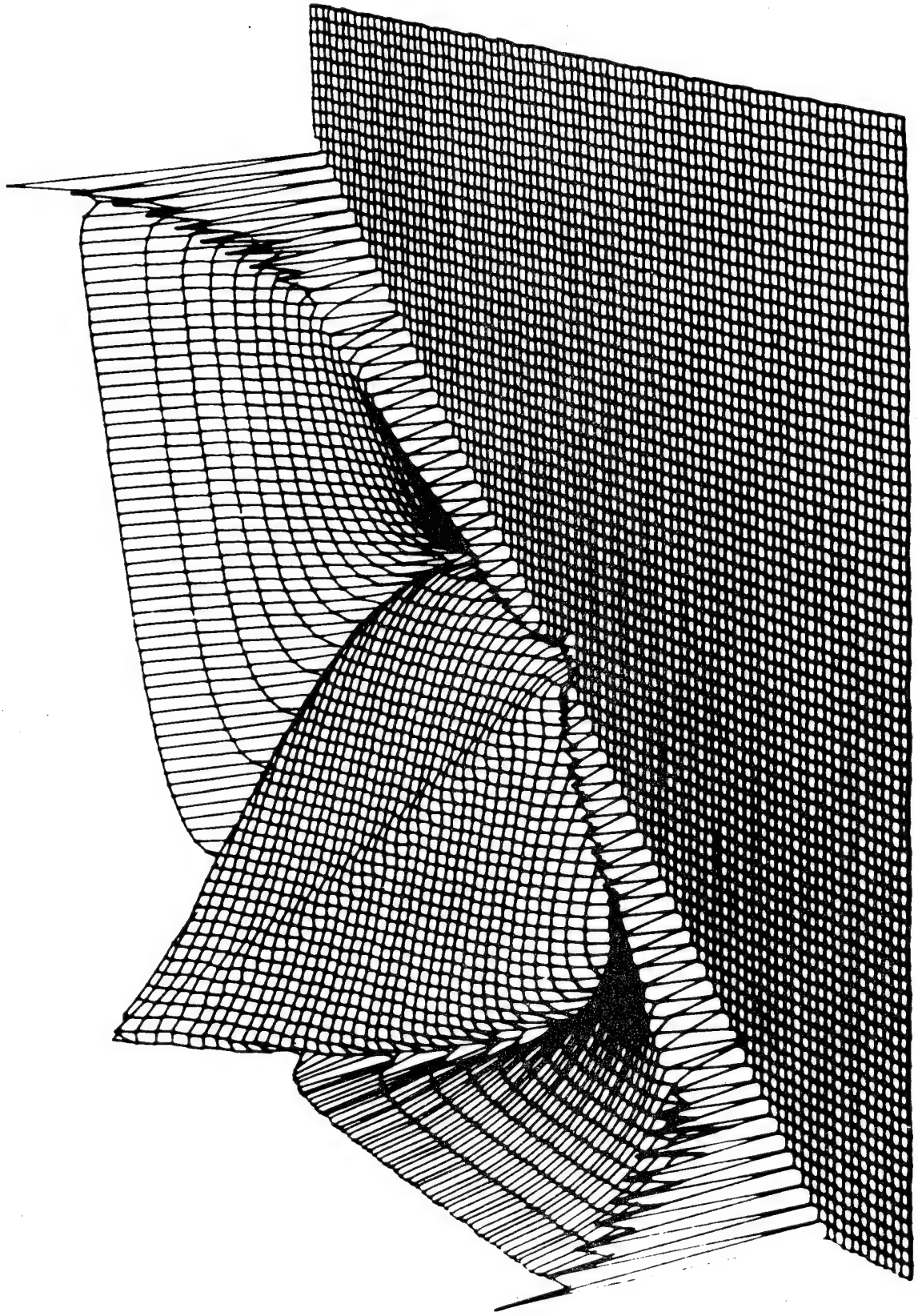

Figure 2b

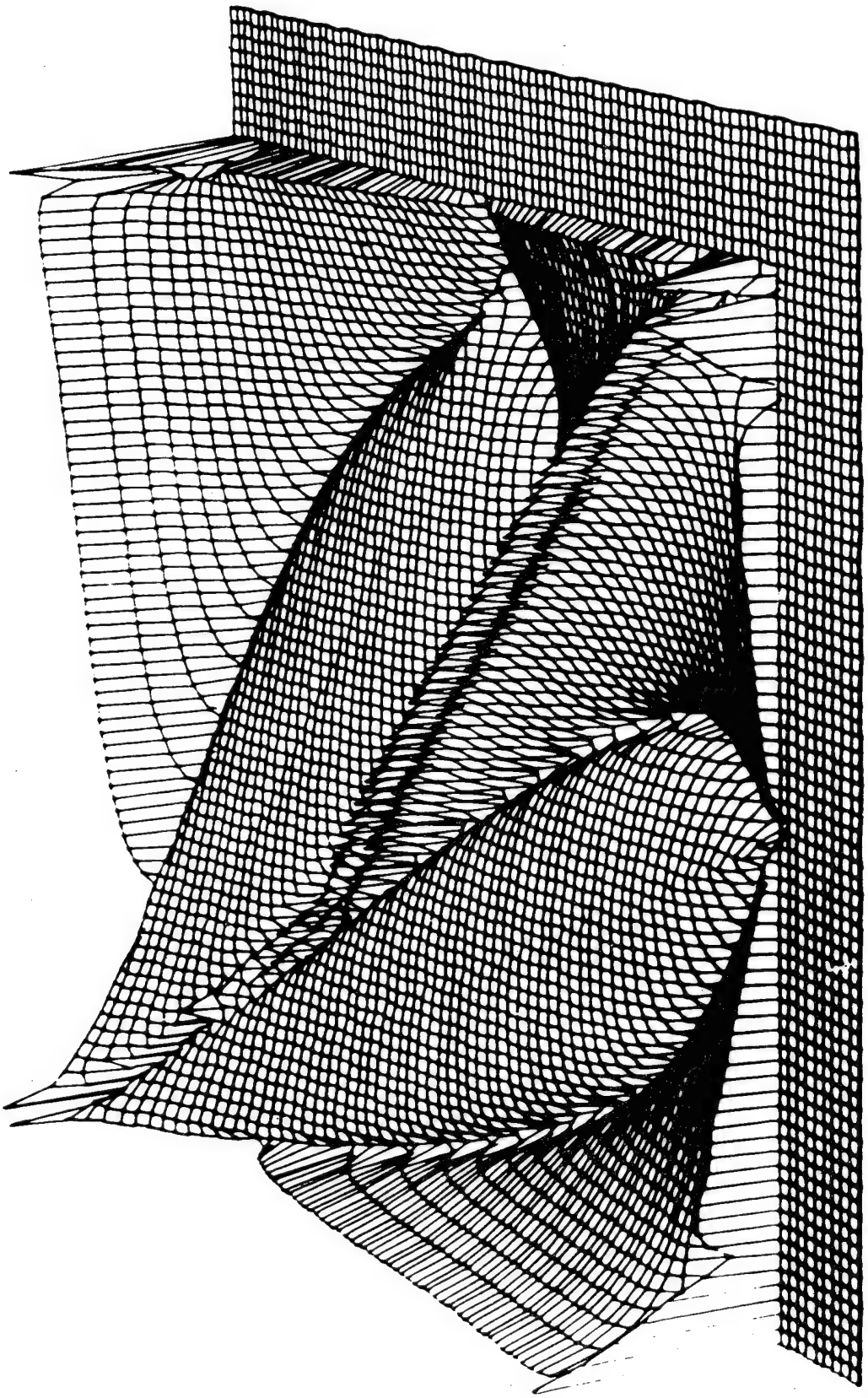

Figure 2c

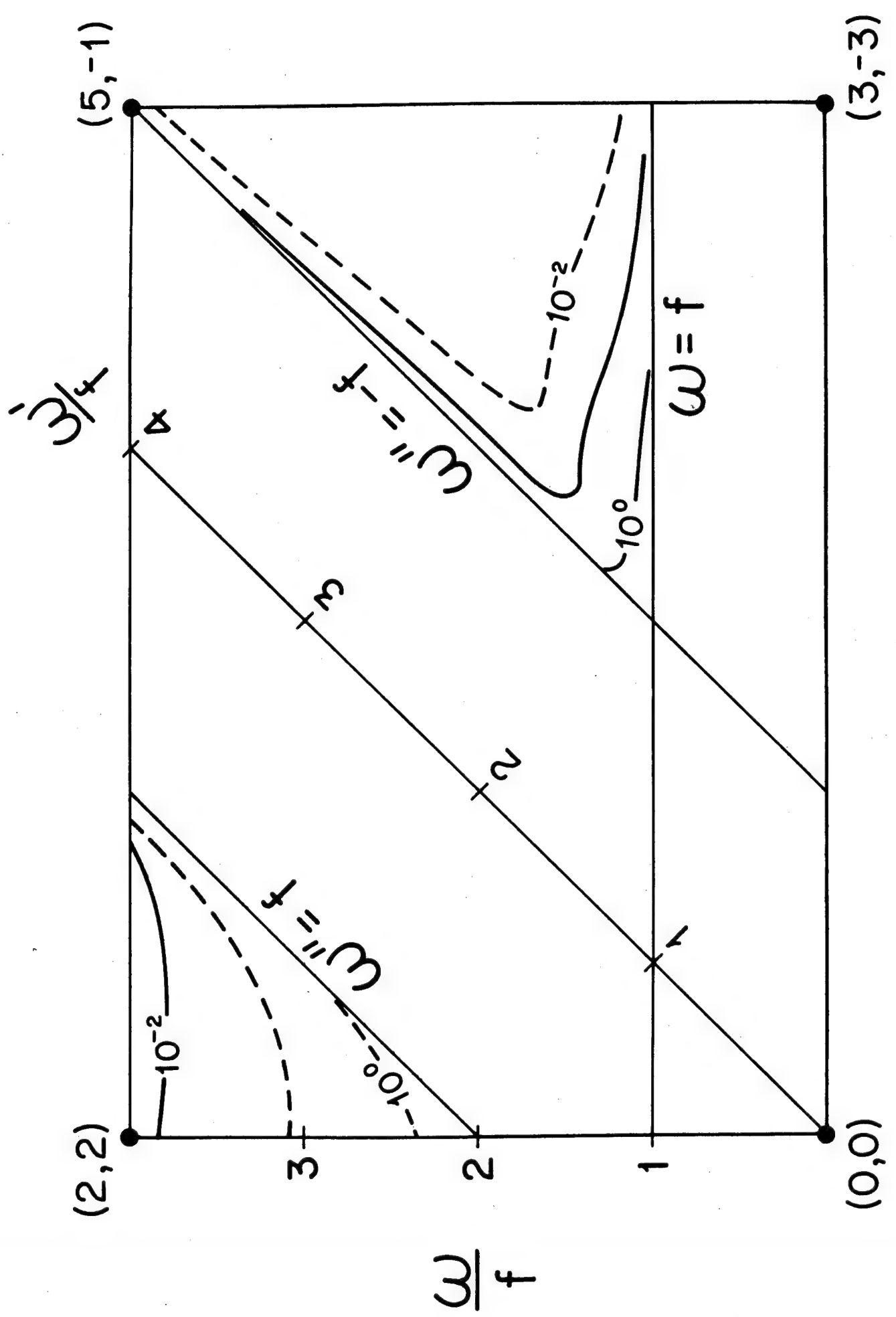

Figure 3

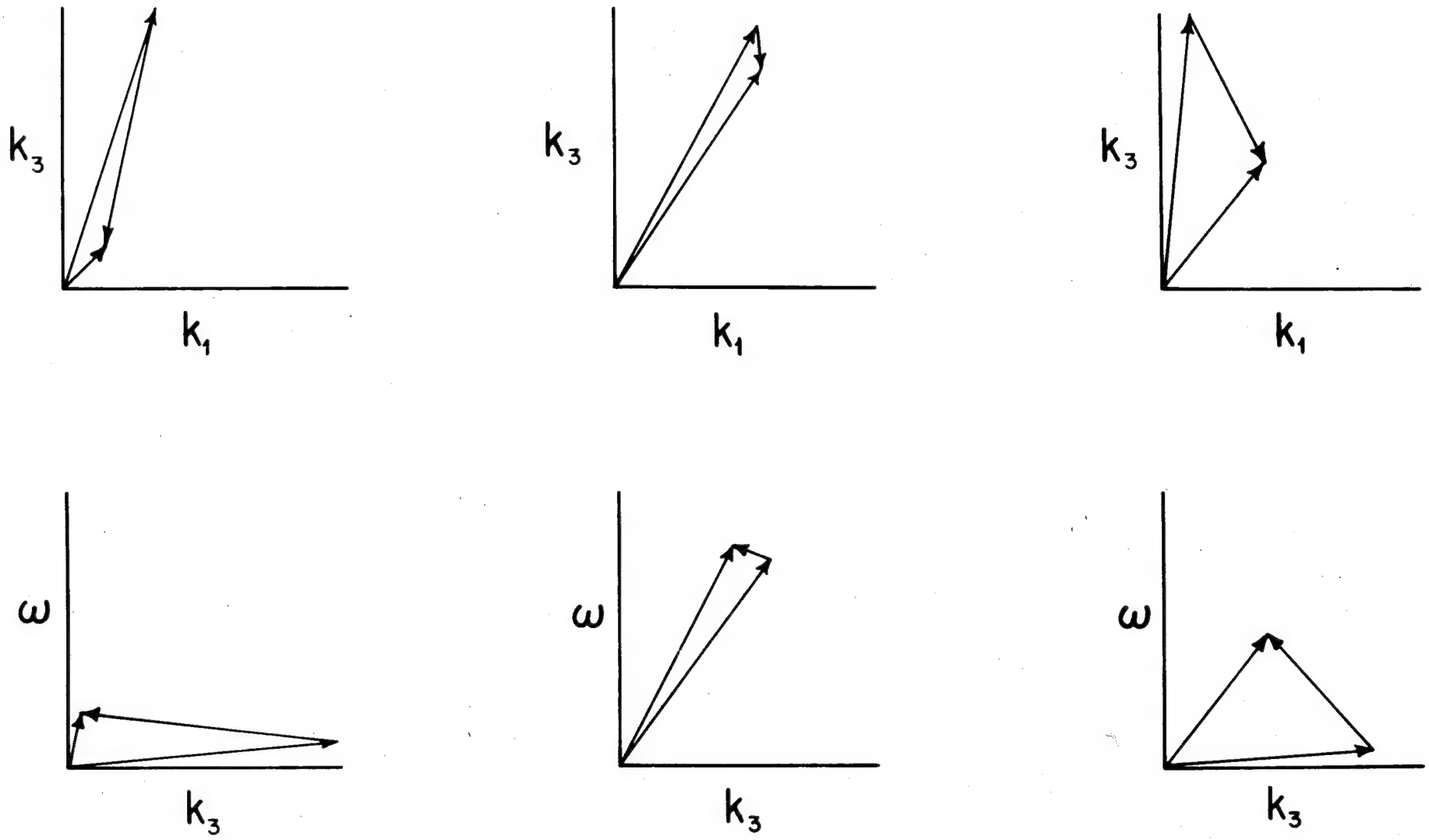

Figure 4

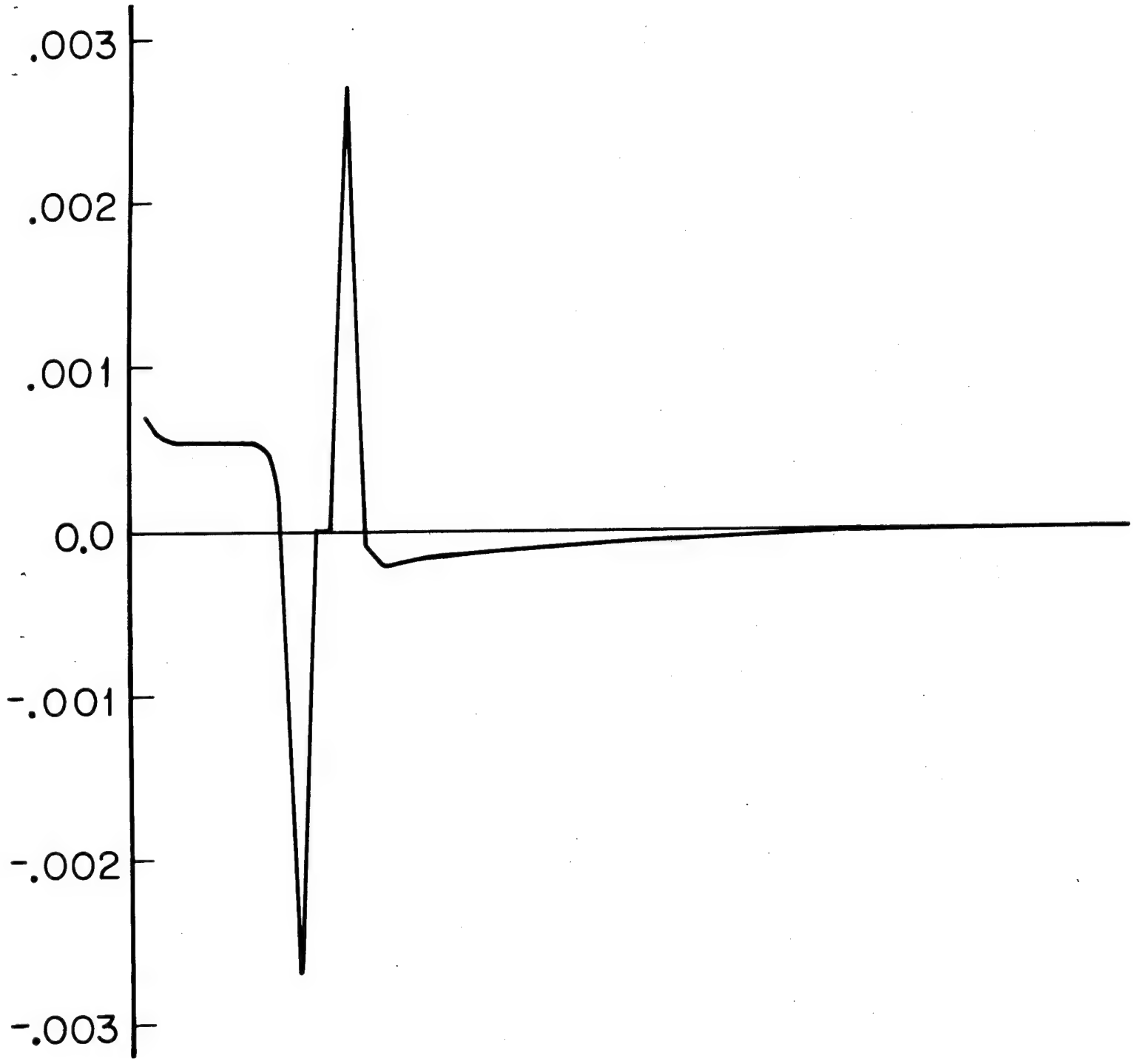

Figure 5a

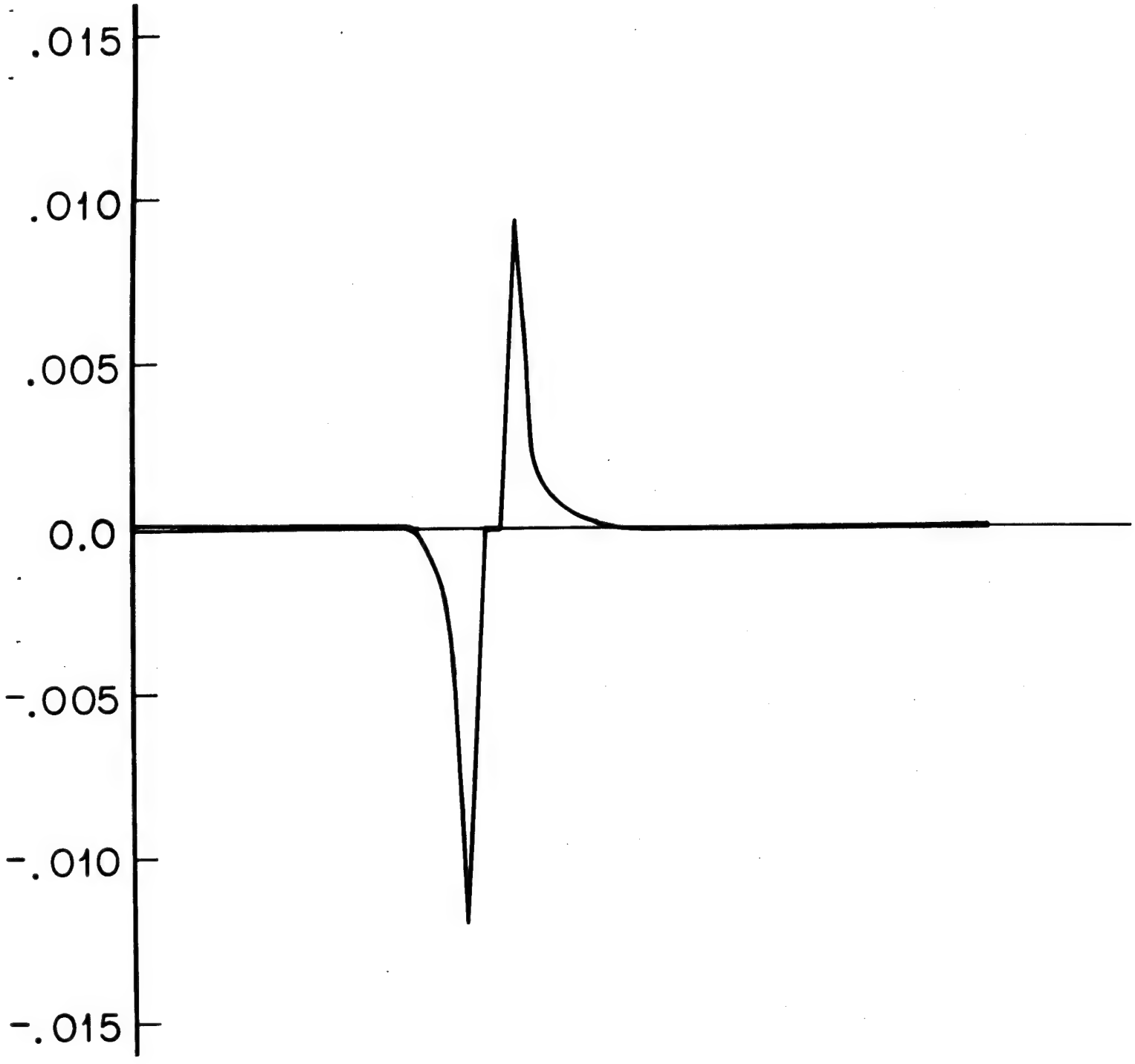

Figure 5b

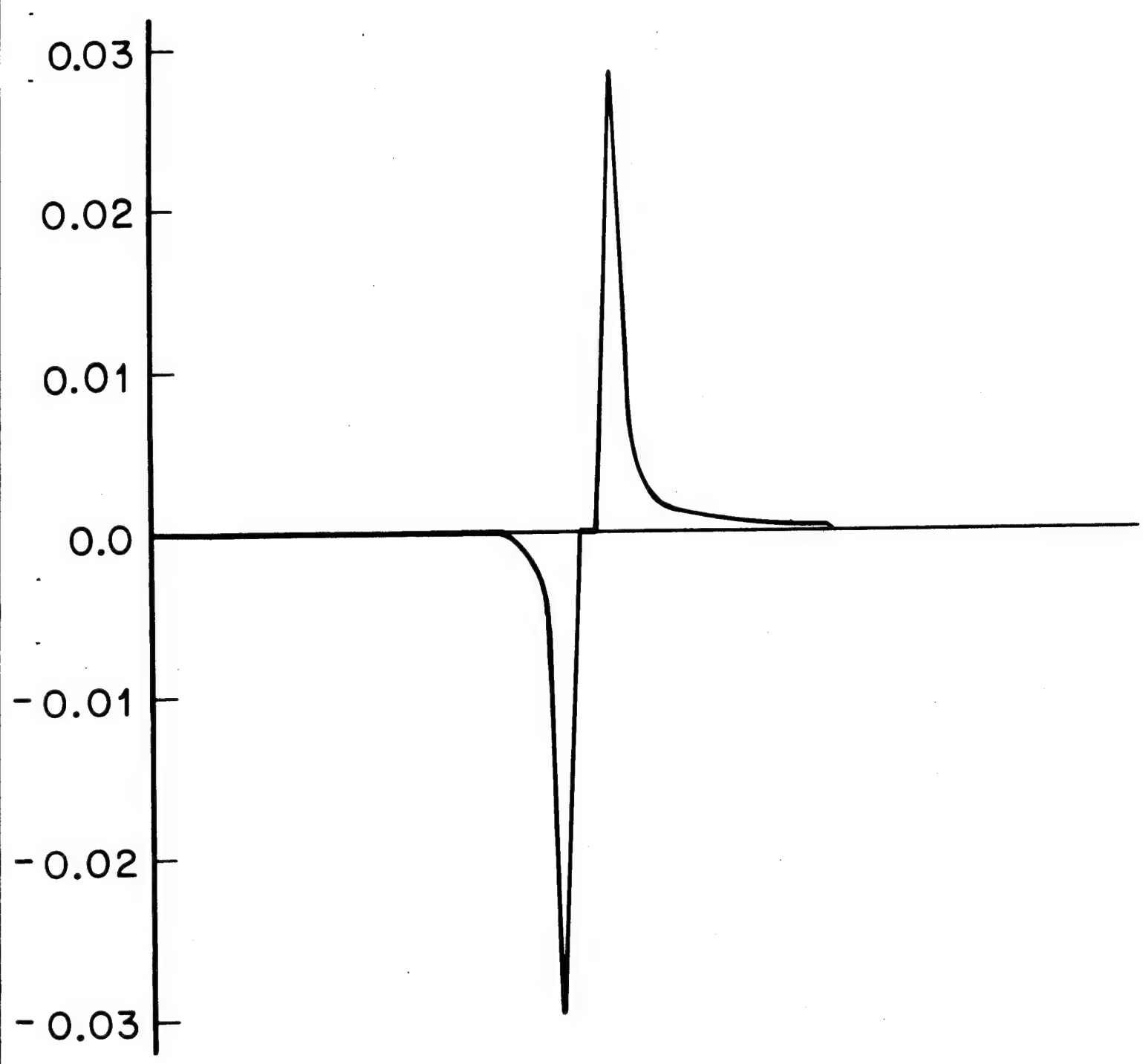

Figure 5c

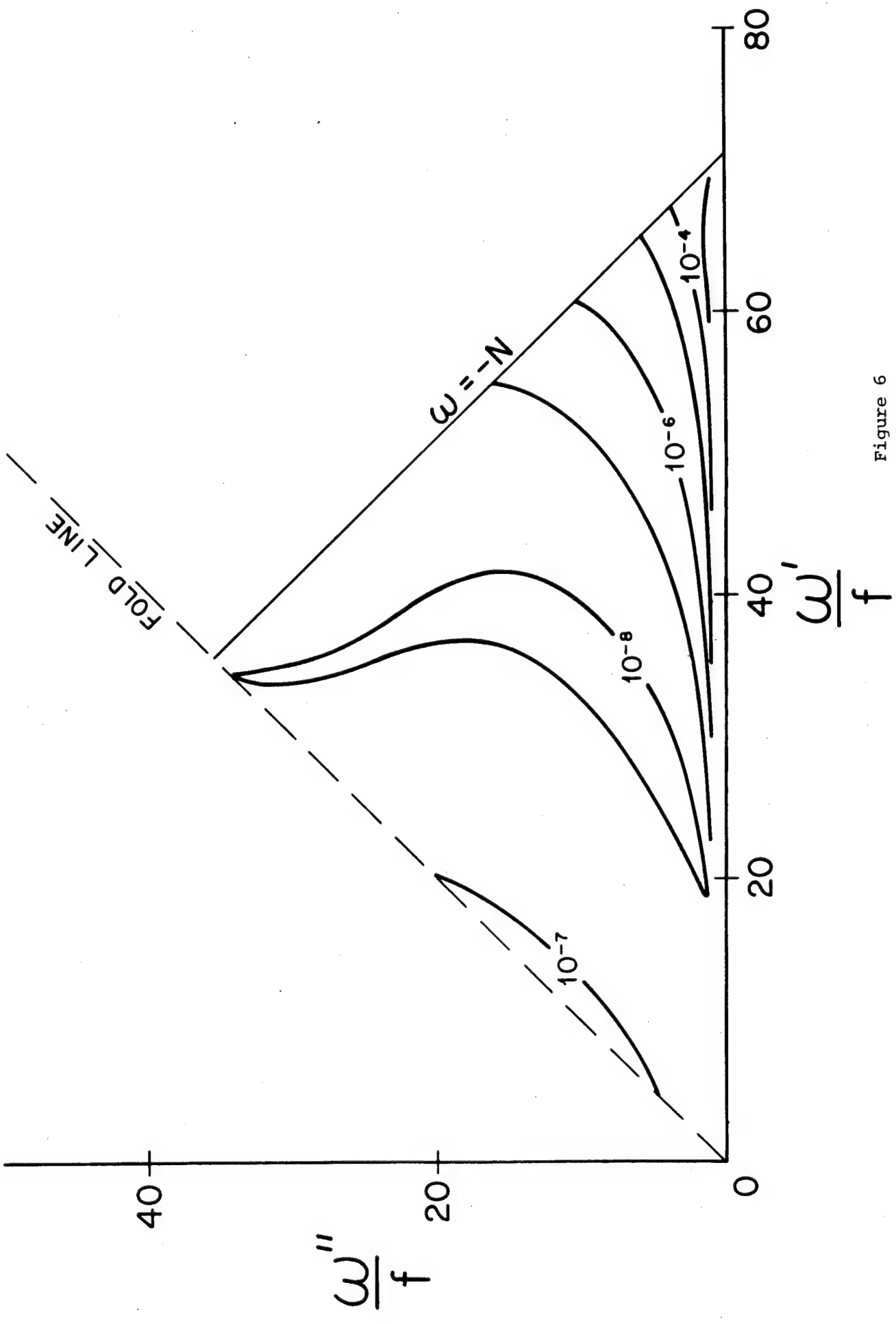

Figure 6

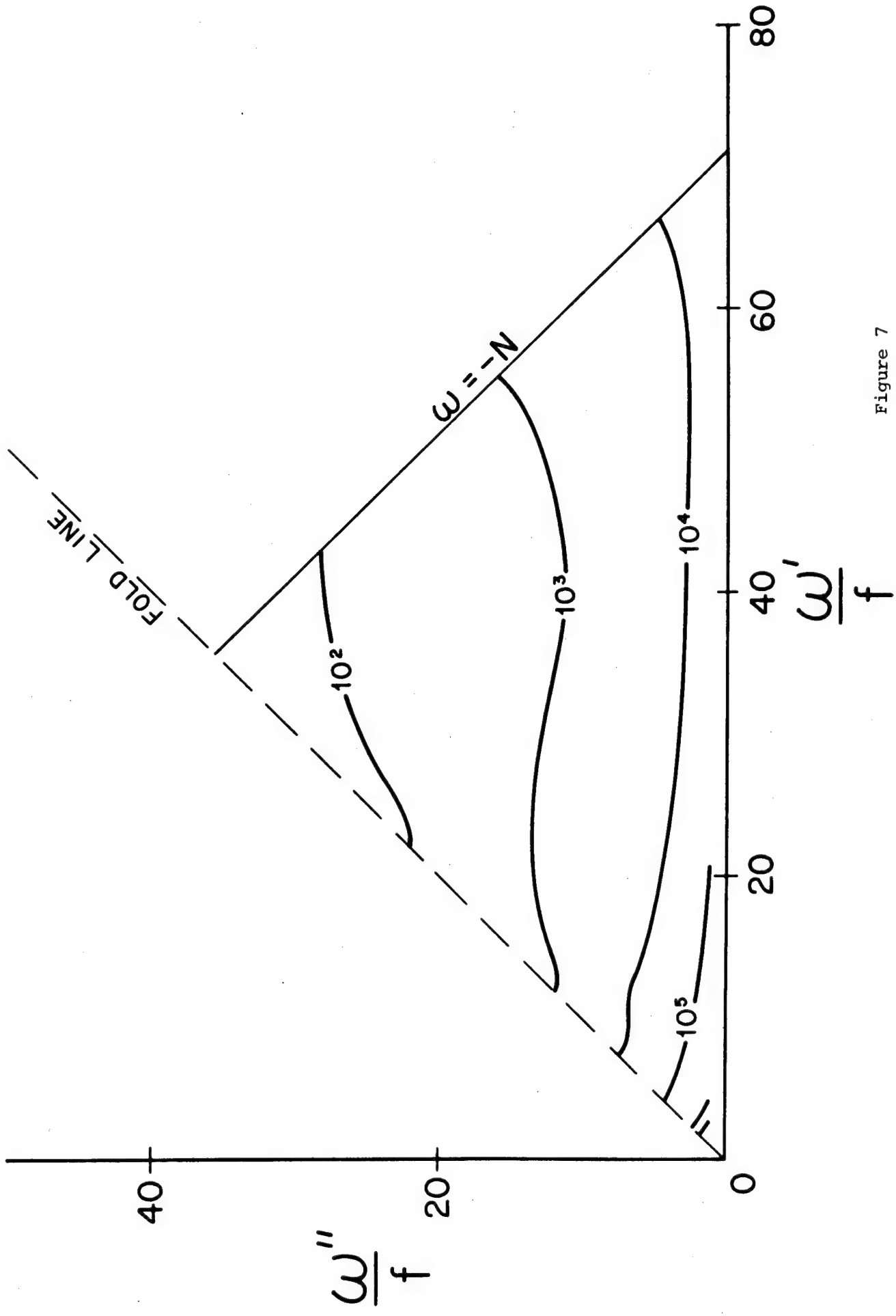

Figure 7

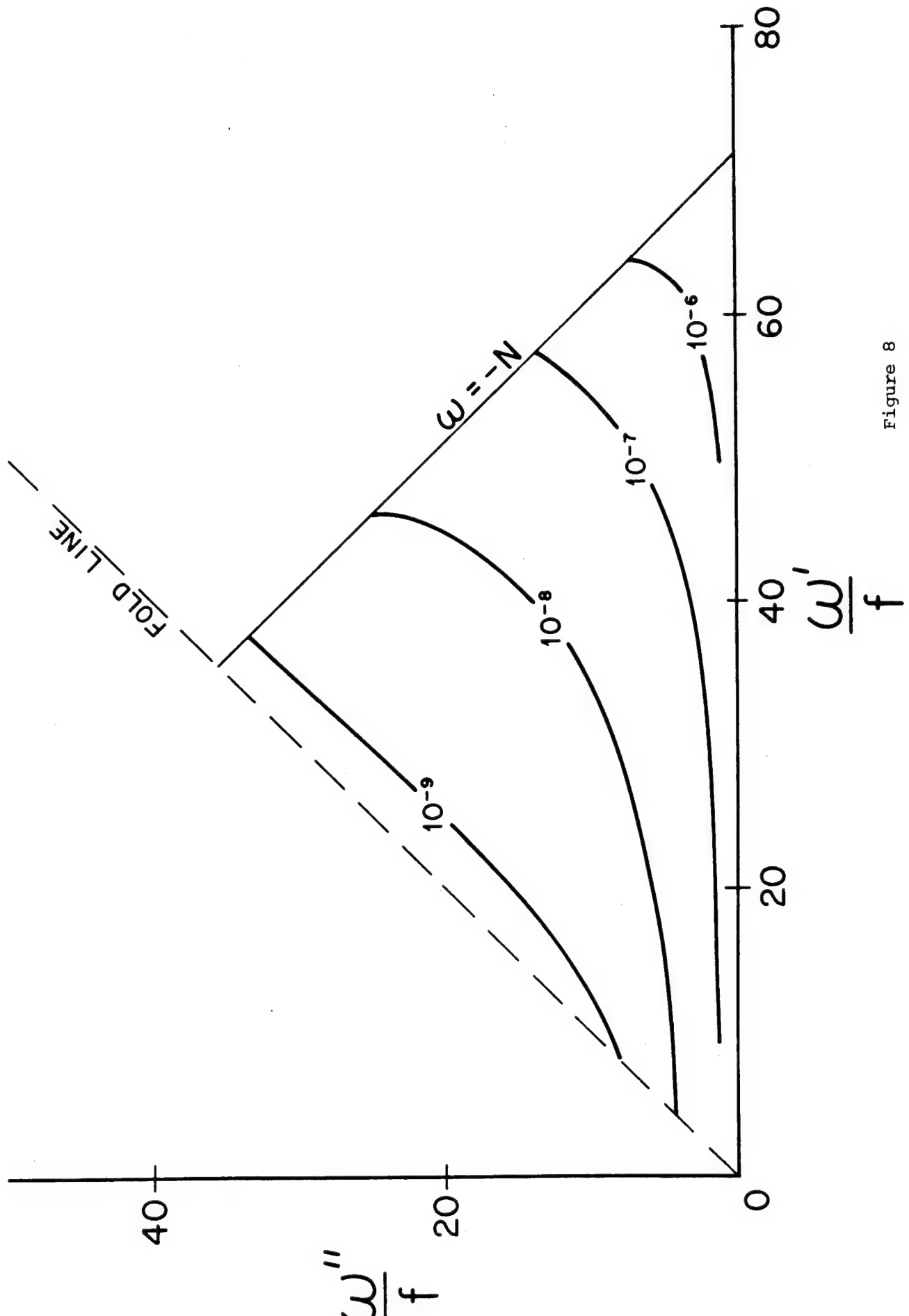
80
60
40
20
0
$\frac{\omega'}{f}$
40
20
$\frac{\omega''}{f}$
$\omega = -N$
FOLD LINE
10^{-6}
10^{-7}
10^{-8}
10^{-9}

Figure 8

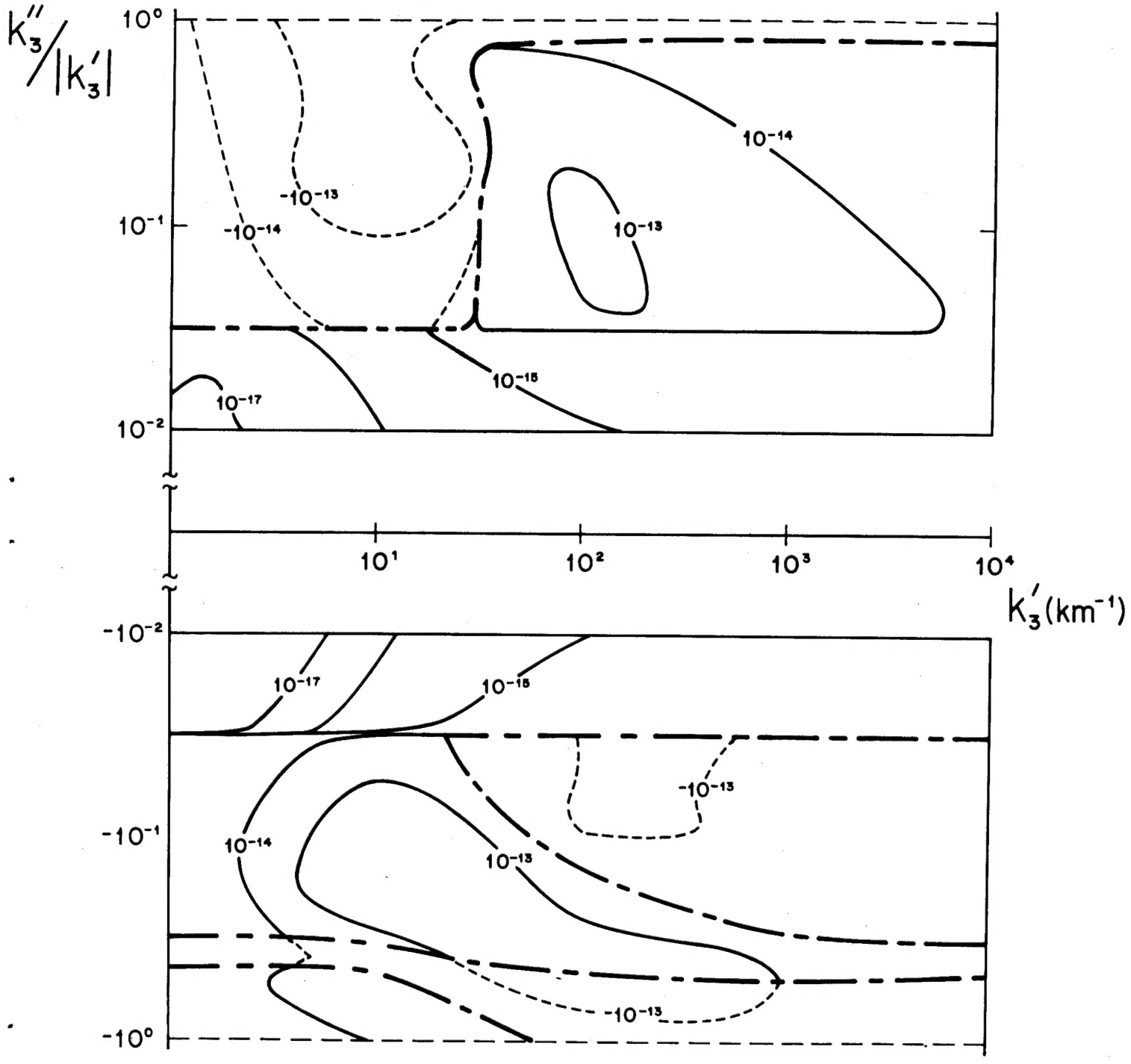
$k''_3/|k'_3|$
10^0
10^{-1}
10^{-2}
-10^{-13}
10^{-14}
10^{-13}
10^{-14}
10^{-15}
10^{-17}
10^1
10^2
10^3
10^4
$k'_3(km^{-1})$
-10^{-2}
-10^{-1}
-10^0
10^{-17}
10^{-15}
-10^{-13}
10^{-14}
10^{-13}
-10^{-13}

Figure 9

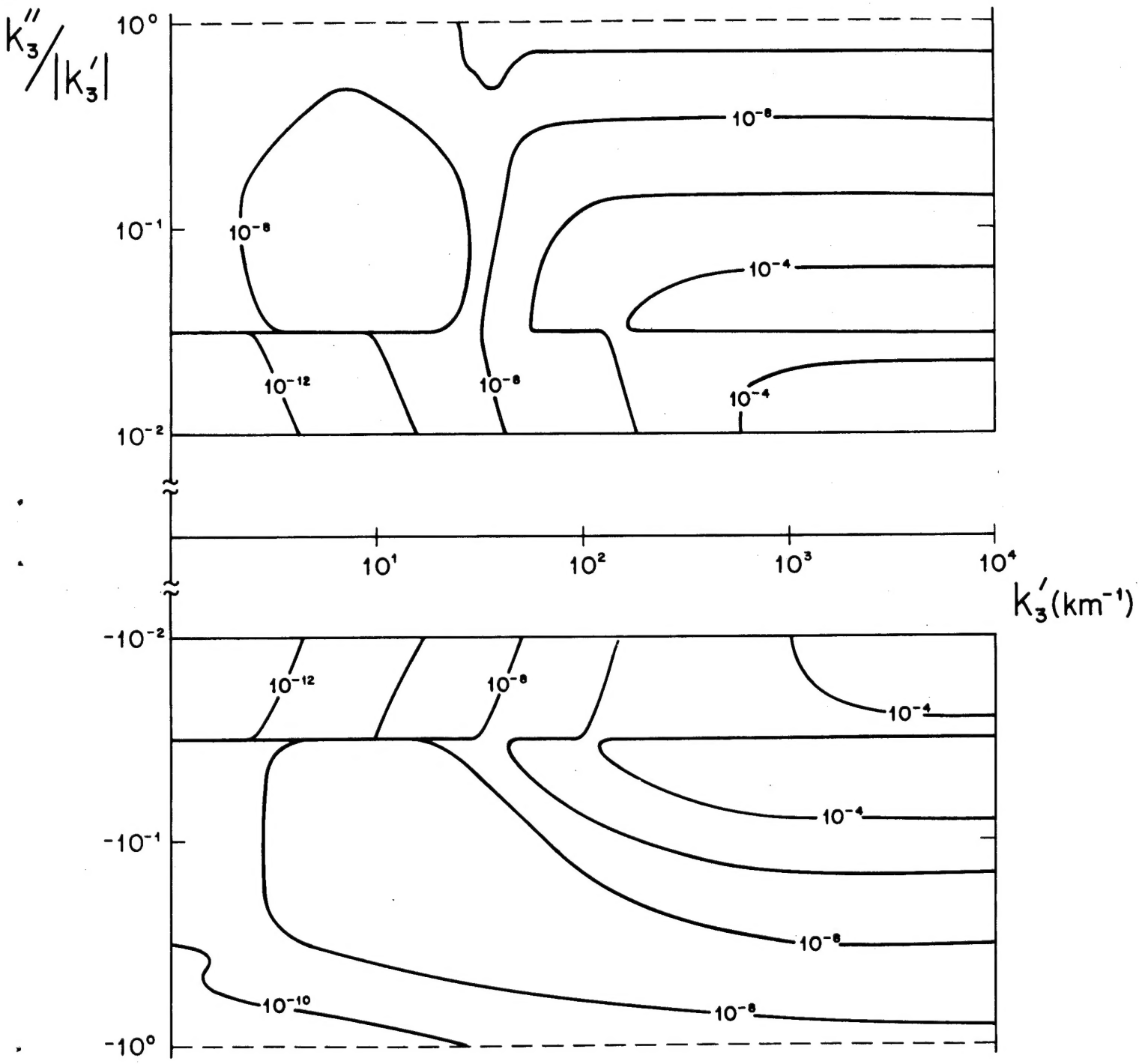
$k_3''/|k_3'|$
10^0
10^{-1}
10^{-2}
10^{-6}
10^{-6}
10^{-4}
10^{-12}
10^{-6}
10^{-4}
10^1
10^2
10^3
10^4
$k_3'(km^{-1})$
-10^{-2}
-10^{-1}
-10^0
10^{-12}
10^{-6}
10^{-4}
10^{-4}
10^{-6}
10^{-10}
10^{-6}

Figure 10

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15. Supplementary Notes

16. Abstracts

Derived from the equations of motion, the bispectrum of power indicates the rate of energy transfer among components of the internal wave field. This, or any other bispectrum, can be evaluated from weak resonant interaction theory given the wave spectrum. Using the Garrett and Munk model of the deep open ocean internal wave spectrum, the bispectrum of power and the closely related auto-bispectrum of vertical displacements are evaluated numerically with the intention of providing an observational test of the weak interaction theory and its predictions. The resulting levels of the bispectra for typical deep ocean internal waves are generally too low to be observed with any statistical confidence in an experiment of reasonable length and cost.

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2. Internal waves
3. Resonant interaction

17b. Identifiers/Open-Ended Terms

17c. COSATI Field/Group

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2. Internal Waves
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